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Ideas about Symbolism That Students Bring to Algebra

Kaye Stacey and Mollie MacGregor

Beginning to learn algebra should be easy. To get started, students need to know that letters of the alphabet are used to stand for numbers and that "answers" often have operation signs in them. If we add 5 to x, we get $x + 5$; if we take away 5 from x, we get $x - 5$. So far so good. Algebra may be puzzling, but it is easy; we do not even have to "work out" the answers. Then comes a stumbling block. The teacher says that $5x$ means 5 multiplied by x, whereas students might think that it signifies addition, like 5 1/2, or place value, like 53.

Algebra starts to get tough. It often cannot say what we want it to say. For example, we can represent "y is more than x" as $y > x$, but we cannot represent the statement "y is 4 more than x" in a parallel way. We must make inferences from the unequal situation just described to write such equalities as $y = x + 4$ or $y - 4 = x$. As a second example, the natural way to describe a number pattern like 2, 5, 8, 11, 14, … is to focus on the repeated addition, perhaps saying, "Start at 2 and keep on adding 3." However, the algebra that most students are first taught cannot express this easy idea in any simple way. To construct the required formula, $y = 3n - 1$, students have to look at the relationship between each number and its position in the sequence. Algebra is a special language with its own conventions. Mathematical ideas often need to be reformulated before they can be represented as algebraic statements.

Teachers may think that students come fresh to algebra, not considering that they already have ideas about the uses of letters and other signs in familiar contexts. Our research with more than 2000 students aged 11 to 15 has uncovered these ideas. By talking to some students, and analyzing the written work of others, we were able to understand why they interpreted algebra as they did.

Their explanations made us see some of the reasons that algebra is hard for beginners. In this article we discuss the following causes of students' common misunderstandings:

- Students' interpretations of algebraic symbolism are based on other experiences that are not helpful.
- The use of letters in algebra is not the same as their use in other contexts.
- The grammatical rules of algebra are not the same as ordinary language rules.

Kaye Stacey, k.stacey@edfac.unimelb.edu.au, teaches at the University of Melbourne, Parkville, Victoria 3052, Australia. Her interests include curriculum issues and students' mathematical thinking. Mollie MacGregor, m.macgregor@edfac.unimelb.edu.au, is a researcher at the same university. She works with teachers and students to improve the teaching and learning of algebra.

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- Algebra cannot say a lot of the things that students want it to say.

Algebra in the Mathematics Curriculum

National curriculum documents in Australia, as elsewhere, promote the view that algebraic thinking begins to develop in the primary grades when children become aware of general relationships in arithmetic procedures, spatial patterns, and number sequences. Australian students are said to "begin algebra" when they learn to write expressions using letters to stand for numbers in generalized arithmetic and as unknowns in simple problems. This process begins when they are about eleven to twelve years old, usually in the first year of secondary school. For all six years of secondary school, algebra units are included in an integrated spiral-mathematics curriculum.

Some Findings from Research

In a test of simple arithmetic and algebra items, one of the questions we asked was the following:

> David is 10 cm taller than Con. Con is *h* cm tall. What can you write for David's height?

Success rates were much lower than we had expected and ranged from 50 percent for students in their first year of algebra learning up to 75 percent for students in their third or fourth year of algebra learning. The great variety of answers showed that students had used or interpreted letters in many different ways. Some of the most common responses, and the assumed reasons for them, are shown in table 1 and are discussed subsequently. We have derived these assumed reasons from previously reported research findings (e.g., Kuchemann [1981]; Pegg and Redden [1990]), interviews with individual students (MacGregor and Stacey 1993), and students' informal working or written explanations on test papers, as in this example:

a	*b*	*c*	*d*	*e*	*f*	*g*	*h*
1	2	3	4	5	6	7	8

These responses were written by students at all levels, including some who were in their third year of algebra learning. If they had been written by people who had never come across algebra, we would say that they were sensible guesses. However, the students we tested had been exposed to an integrated secondary mathematics curriculum that included some algebra units each year. It seemed that some of them, even after three years, did not know what the letters represent in algebra, that is, generalized numbers, unknown numbers, or variables.

Table 1
Responses to the "David's Height" Question

Response	Assumed Reason
D*h*	This abbreviation stands for the words "David's height."
C + 10 = D	The C means "Con's height," and the D means "David's height."
$h = h + 10$	The *h* denotes the concept of height, so that it can refer to both people's heights in this problem.
18	Since H is the eighth letter of the alphabet, 8 + 10 = 18.
R	The tenth letter after H in the alphabet is R.
160	Think of a reasonable height for Con (e.g., 150 cm); add 10.
11	Any letter equals 1 unless specified otherwise; therefore, $10 + h = 10 + 1 = 11$.
*h*10 or 10*h*	A letter next to a numeral indicates addition.
*h*10	The 10 on the "positive," or right-hand, side of *h* means "10 more."

Students' Inventions Reflect History

Students' first attempts to interpret and use algebraic letters are usually based on sensible reasoning and draw on a range of previous experiences. They are often thoughtful attempts to make sense of a new notation. We found that most children in a class of eleven-year-olds who had never been taught algebra guessed that letters stood for abbreviated words—D for David and *h* for height—or for specific numbers. These specific numbers were either the "alphabetical value" of the letter, for example, $h = 8$, so $h + 10 = 18$, or an arbitrary but reasonable value of the quantity described in the problem, for example, 150 cm is a reasonable height for Con, so David's height would be 160 cm.

Interpreting *a* as 1, *b* as 2, and so on, which leads to the answers 18 and R shown in the table, has its parallel in the early Greek numeration system in which each number was denoted by a letter. It is an interpretation that is often used today in puzzles and secret-code games. We know of some textbooks that make extensive use of alphabetical codes in answer keys for "self-correcting" homework assignments. Teachers need to address explicitly the confusions that may arise from these practices. Another sensible guess that has its roots in history is the use of conjoining and ordering, for example, *h*10 means 10 more than *h* and 10*h* means 10 less than *h*. This notation has its parallel in the Roman numerals VI for "one more than five," IV for "one less than five," and so on. In the Roman numeration system, symbols placed together indicate addition and subtraction, not multiplication as in algebra. It is an analogue of the physical process of joining objects, which models the addition of quantities.

The Right Meaning in the Right Context

Students are told that in algebra, letters stand for numbers. However, they see letters used with other meanings. Letters are used in many contexts, both within and outside mathematics, as abbreviated words or as labels: "p. 6" means "page 6"; cm means "centimeters"; and $\angle ABC$ labels an angle in a geometric figure, with the letters *A, B,* and *C* denoting positions or points. Quantities are frequently denoted by the initial letters of their names. Teachers talk about *m* as the "mass" and *t* as the "time taken"; they make statements like "Let *C* denote the circumference" and "We'll use *C* to stand for the cost."

Although teachers know that the letters *m, t,* and *C* denote numbers of units of measure and not words, some students see them as standing for the words themselves. It is no surprise that many of the students we tested wrote D*h*, D, or *h* to mean "David's height." Since height is a numerical concept, the word *height* and the quantity "number of cm tall" can be synonymous when thoughtfully used. However, responses from many students showed that they used *h* loosely, often as a particular attribute of the person Con or as the general word *height*. The following examples are from year-10 classes:

> $h = +\ 10$ (This example is an attempt to translate "the height is 10 more.")
>
> $h = D + 10$ (Con's height is *h;* David's height, D, is 10 more.)
>
> $Dh = h + 10$ (D*h* is David's height, *h* is Con's height.)
>
> *h* = David – 10 (The *h* means "Con's height," but since no symbol has been given for "David's height," the word "David" is used.)

New Information Misinterpreted

Some older students made errors in writing totals and products that were not made by younger students. They wrote *h*10, meaning "*h* plus 10," for David's height, assuming that conjoining meant addition. When they had to write "*x* times 4," for another item in the test, they wrote x^4. Younger students did not make this mistake because they had not learned the notation for powers. When we talked to some fifteen-year-old students, we found that they thought of exponents as an instruction to multiply, without having a clear idea of what was being multiplied.

The belief that any letter alone stands for 1 was another obstacle for older students. Students explained that "by itself the letter is one thing, 1" and that "*x* is just like 1, like having one number." One likely cause of this belief is a misunderstanding of what teachers mean when they say "*x* without a coefficient means 1*x*." The student gets a vague message that the letter *x* by itself is something to do with 1. Other sources of confusion for older students are the facts that the power of *x* is 1 if no index is written ($x = x^1$) and that $x^0 = 1$.

Interference from Prior Knowledge

Students' interpretation of equations can be influenced by prior experiences in arithmetic. Their background of arithmetic has been built on a foundation in which the equals sign means "gives" or "makes," as in "3 plus 5 gives 8." Teachers see evidence of this interpretation when students working multistep calculations frequently use the equals sign for partial answers, moving from left to right, as in $3 + 5 = 8 \times 7 = 56 \div 2 = 28$. This restricted but familiar use for the equals sign is an obstacle to understanding equations.

We gave students the equation $a = 28 + b$ and asked them to decide which of the following would be true:

(i) a is greater than b.

(ii) b is greater than a.

(iii) $a = 28$.

(iv) You cannot tell which number is greater.

About one-quarter of students at all levels made incorrect choices, and in several schools, the percent wrong was far greater. Some students thought that since the letters stood for unknown numbers, they could not tell which was greater: "They could be anything." Some thought that b was greater because it had 28 added to it, whereas a had nothing added to it. Some thought that a equaled 28 because the equation said "a equals 28, then add b."

Interference from Natural-Language Rules

Interpreting $a = 28 + b$ to mean "a equals 28, then add b" arises from reading the equation like ordinary English. In a simple statement of English, the events described occur in the stated order unless some change of order is specially signaled. For example, when we read the instruction "Enclose your check and seal the envelope," we know that the first thing to do is to put the check in the envelope. The same instruction can be expressed as "Before sealing the envelope, make sure your check is enclosed." The word *before* signals that sealing the envelope, although described first in the sentence, is not the first thing to do. In a mathematical equation, the signals for ordering are not those of ordinary language. They include parentheses (which are not used as we are using them here in the ordinary language way) and more subtle signals that must be deduced from a knowledge of formal rules for the precedence of operations. Natural-language rules are no help in reading mathematical expressions.

Algebra's Limited "Vocabulary"

Another obstacle arising from a false analogy with ordinary language is students' expectation that any procedures they can think about or talk about can be written in simple algebra. Consequently they have difficulty generating formulas from number patterns and tables. It is well known that students tend to look for rules for calculating the next number in a sequence instead of rules relating two variables. Figure 1 shows an item requiring students to write such a rule.

Most students we tested could use the chart in figure 1 to calculate further values, showing that they recognized the rule, but they could not write an equation. Some tried to invent notations for expressing their ideas, such as $x + 4y$ to mean "Start with x and add 4 to get y"; $x \uparrow 1$, $y \uparrow 1$ to mean "As x goes up by 1, y goes up by 1"; $1x = 5y$ to mean "When x is 1, y is 5"; and $x = 3y$ to mean "There are three numbers between x and y." Teachers need to explain that algebra is a restricted language. It cannot be used to write many of the things that ordinary language can say. Students need to think about the concepts and relationships that they want to express and decide how to restructure them into a usable form for algebra.

x	y
1	5
2	6
3	7
4	8
5	9
6	___
7	11
8	___
.	.
.	.
.	.

(i) When x is 2, what is y?

(ii) When x is 8, what is y?

(iii) When x is 800, what is y?

(iv) Explain in words how to work out y if you are told what x is.

(v) Use algebra symbols to write a rule connecting x and y

Fig. 1. Item to test for recognizing a function and writing an equation

Summary

We have shown how students bring a variety of experiences to their interpretation of beginning algebra. These experiences include the following:

- the many uses of letters in other contexts; operations implied in composite symbols, such as 5 1/2, 53, and VIII;
- reading the equals sign as "makes" or "gives" and using it to link parts of a calculation; and
- features of natural language, such as indicators of temporal sequence, that students assume carry over into the formal language of algebra.

We have also pointed out that new learning, such as the concept of powers and its notation, can destabilize old knowledge that is not secure.

It is natural and healthy that students interpret new ideas in terms of prior experiences. The ways in which complete novices in our sample interpreted algebraic language showed common sense and initiative. However, it is disturbing that some students who had spent a considerable time in algebra classes still interpreted algebraic expressions incorrectly. Teachers who recognize the many sources of misunderstanding and point them out in their teaching can improve students' performance. For example, when two teachers we know were shown the effect on their classes of the belief that A = 1, B = 2, and so on, they were able to correct this misunderstanding easily and quickly. Teachers need to help students appreciate that algebra is a special language that has its own conventions and uses familiar symbolism in new ways.

The following suggestions are offered to help teachers meet the challenges of dealing with the prior knowledge that students bring to their study of algebra.

- Use algebraic notation more often. Use it when revising and extending students' knowledge of arithmetic; spread it through other topics in the mathematics course as a useful and precise language for generalizing and for writing formulas.
- Emphasize that letters in algebraic expressions stand for numbers, not for names of things. Do not say, for example, "We'll use *c* to stand for the cost" but instead say "We'll use *c* to stand for the number of dollars." It is often hard to use simple English while stressing that the letter represents a number. For example, it is tempting to say "Let *L* stand for the length" rather than the more complex statements "Let *L* stand for the number of meters in the length" or "Let *L* stand for the number of meters long." Teachers can compromise by saying, "Let the length be *L* meters" or "Let *L* be the length in meters," but when doing so, they should stress that *L* stands for a number and not for the word *length*.
- Check that your students clearly distinguish products and powers in arithmetic and the ways of writing them in arithmetic and in algebra. Most teachers explain that a^b means "*a* multiplied by itself *b* times," a more accessible phrase than the more rigorous "the product of *b* factors, each factor having the value of *a*." However, "*a* multiplied by itself *b* times," as well as being wrong or ambiguous, is easily conflated with "*a* multiplied by *b*."
- When starting work on number patterns and functions, ask students to explain in words the relationships they see. Discuss why some of their verbal descriptions can be written as equations and some cannot. Help them restructure their descriptions so that they can be translated to algebra.
- Make sure that your students do not think that letters have specific values that depend on their position in the alphabet. Be wary of games and puzzles that might promote this belief.
- Appreciate that students come to algebra with rich prior experiences of symbol systems. Help them sort out which of these experiences interfere with their algebra learning and which support it.

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