Learning Mathematical Symbolism: Challenges and Instructional Strategies

Rheta N. Rubenstein and Denisse R. Thompson

> ***It is often said that mathematics is a symbolic language . . . . The symbols of mathematics, like the letters or characters in other languages, form the written language of mathematics.***
>
> —Zalman Usiskin, "Mathematics as a Language"

Do you know students who—

- get -3 when evaluating $-x$ for $x = -3$?
- read r^2 as "*r*-two" or $\log_2 8$ as "log of two to the eighth"?
- think $f^{-1}(x)$ means

$$\frac{1}{f(x)}?$$

Symbolism is one of the hallmarks of mathematics. As Usiskin (1996) notes, mathematical symbols are the means by which we write mathematics and communicate mathematical meaning. Pimm considers the symbolic feature of mathematics to be "one of the subject's most apparent and distinctive features" (1991, p. 19). Indeed, he indicates several functions performed by symbols: they illustrate the structure of mathematics, help make manipulations routine, enable reflection about mathematics, and facilitate compactness and permanence of thought.

The symbolic language of mathematics often challenges our students. We sometimes forget that the words, phrases, and symbols that are meaningful to us are unfamiliar to students. As a result, many students have difficulty verbalizing, reading, understanding, and writing mathematics to express their mathematical thoughts, reflect on concepts, or extend ideas. As we strive to offer rich mathematical experiences that help students make sense of mathematics and become confident problem solvers, we must still attend to a long-standing objective: building students' fluency with conventional mathematical symbolism. Students who cannot communicate by using standard symbolism will at some point be hindered in their mathematical development.

The purpose of this article is to sensitize high school and college teachers to problems, or challenges, that students often have with mathematical symbols and to suggest instructional strategies that can reduce such difficulties. Using symbols fluently is a necessary, albeit not sufficient, condition for overall mathematics achievement. We begin by discussing various uses of symbols, then identify common difficulties encountered as students verbalize, read, and write symbols. We conclude by offering teaching strategies that can avoid or overcome these difficulties. Throughout, we draw on work by Kane, Byrne, and Hater (1974); Shuard and Rothery (1984); and Reehm and Long (1996), combining their earlier analyses with our own extensions.

USES

To facilitate our discussion, we first consider the many ways that symbols are used. **Table 1** shows six categories of uses, as well as examples from five basic strands of mathematics. Although this list is not exhaustive, it gives us a framework for discussing issues of symbolization. As row 1 indicates, symbols are certainly used to name such concepts as numbers, geometric shapes, and functions. To draw an analogy to language, these naming uses are *nouns.* Row 2 illustrates the use of symbols to show relationships between concepts. To continue the language analogy, such relationships as "is equal to" or "is a subset of" play the role of *verbs.* Stating a relationship creates a complete mathematical *sentence.*

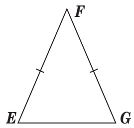
Symbols facilitate compactness and permanence of thought

Rows 3 and 4 illustrate using symbols as *operators.* Those in row 3 are unary operators, that is, they operate on a single input, as in finding the

Rheta Rubenstein, rhetar@aol.com, teaches at Schoolcraft College, Livonia, MI 48152. She is interested in making mathematics meaningful and accessible to all students. Denisse Thompson, thompson@tempest.coedu.usf.edu, teaches at the University of South Florida, Tampa, FL 33620. She is interested in curriculum development, assessment, and the integration of literature into the mathematics classroom.

TABLE 1

Uses of Mathematical Symbols

Use	Examples
1. Name a concept.	Number: 24, 2/3, 0.85, –5, 29%, 4 : 5, π, e, i Algebra and calculus: x, $f(x)$, $(-4, 5]$, ∞ Geometry and measurement: $\angle A$, $\triangle ABC$, $\overline{EF}$ Statistics and probability: x, χ^2 Discrete mathematics: $\varnothing$, $\aleph_0$
2. State a relationship.	Number: $2.3 \neq 2.5$, $3 < 7$, $5.99 \approx 6$, $3 \mid 12$. Algebra and calculus: $2x + x = 3x$. Geometry and measurement: $\triangle ABC \sim \triangle RST$. In isosceles $\triangle EFG$, $\overline{EF} \cong \overline{GF}$ can be shown visually, as well as symbolically.  Statistics and probability: $P(X) < P(Y)$. Discrete mathematics: $A \subset B$
3. Indicate an operation or a function with one input.	Number: –(7), $\lvert -4 \rvert$, 5^{-1} Algebra and calculus: $-x$, $\sqrt{x}$, $[x]$, $\int f(x)\,dx$ Geometry and measurement: Area (circle O) Statistics and probability: $n!$ Discrete mathematics: $[A]^{-1}$, det $[A]$
4. Indicate an operation or function with two or more inputs.	Number: 3 + 7, 4 ^ 3, G.C.D. (12, 18) Algebra and calculus: $f \circ g$, f/g Geometry and measurement: $(1/2)bh$, $2L + 2W$ Statistics and probability: mean $\{x_1, x_2, x_3, \ldots, x_n\}$ Discrete mathematics: $A \cap B$, $q \wedge r$, $C(n, r)$
5. Abbreviate words, units, theorems, and so on.	Number: % Algebra and calculus: ' " (first or second derivative) Geometry and measurement: ¢, in., kg, °, SAS Statistics and probability: s.d. Discrete mathematics: $\Rightarrow$ ("implies"), $\therefore$ ("therefore")
6. Indicate grouping.	Explicit: ( ), [ ], { } Implicit: $\sqrt{9+16}$, log xy, 2^{x+3}, $\dfrac{3+5}{4+4}$

opposite of a value or finding the factorial of a number. The symbols in row 4 operate on two or more inputs, as in adding, exponentiating, or averaging. Extending the language analogy once more, operators joining inputs produce *phrases.*

The calculator helps clarify the distinction between functions of single inputs, or unary operators, from functions of two inputs, or binary operators. For example, students can find the cube of a number by using the exponentiation key, a binary operator, for which they must enter a base and 3 for the exponent. Alternatively, they may use a built-in cubing key, a unary operator, for which they need only provide a base. Calculators also use a language analogy when they display *syntax error* if improper structure is used.

Row 5 illustrates symbols used for *abbreviation.* When students encounter abbreviations, they must think of the concept to which the abbreviation refers. For instance, when students see *s.d.,* they need to think and read *standard deviation.*

The grouping symbols shown in row 6 function as *punctuation.* In mathematical language, grouping symbols may be shown explicitly or may be implied. For example, the understood grouping implied by the radical symbol in $\sqrt{9+16}$ needs to be made explicit with parentheses when entered into a calculator: √(9 + 16). Implied or understood grouping symbols sometimes create difficulties because students fail to make the grouping symbols explicit in situations involving translation or evaluation.

CHALLENGES RELATED TO LEARNING MATHEMATICAL SYMBOLS

Understanding the different uses of symbols shown in **table 1** facilitates analyzing the challenges that arise as students learn to read and use those symbols. The following discussion is divided into three areas. *Verbalization challenges* are those that involve translating symbols into spoken language. *Reading challenges* are those that deal with understanding the concepts represented by the symbols. *Writing difficulties* are those that deal with producing symbols.

The challenges within these three areas do not occur in isolation but often occur simultaneously. Skemp, as noted in Pimm (1987), identified two levels of language: *surface structures,* that is, the written symbols, and *deep structures,* that is, the conceptual meanings. Verbalization relates generally to the surface structures used to transmit ideas; reading and writing symbols involve accessing and using the conceptual meanings.

Challenges in verbalizing symbols

Usiskin (1996, p. 236) noted, "If a student does not know how to read mathematics out loud, it is difficult to register the mathematics." Reading is a link to understanding. Unlike with unfamiliar English words, however, students cannot "sound out" newly learned mathematical symbols. How we pronounce symbols or associated phrases—for example, *divided into* as opposed to *divided by*—constitutes conventional knowledge; novices must be told the convention.

Other issues related to verbalization are shown in **table 2.** Some symbols require multiple words to pronounce (row 1), and others are verbalized in multiple ways (row 2). At times, the verbalization of a symbol changes depending on the context. For example, in arithmetic, 12 – 5 is commonly vocalized as "12 take away 5" or "12 minus 5"; later in algebra, however, $x - y$ is sometimes vocalized as "x less y " or "y less than x." When studying distance on the number line, verbalizing $x - y$ as "the difference" between two coordinates is helpful. Students may need to be reintroduced to verbalizations of familiar symbols when they are doing more advanced work.

Challenges in reading symbols

When we verbalize mathematical symbols, we are operating with their surface structure. The deep structure, or meaning, is closer to the heart of the learning that we seek. **Table 3** identifies issues that are related to the deeper structures of reading and understanding symbols. In general, context is an important guide for meaning. For example, in analyzing functions, the ordered pair (2, 3) may be an extreme *point* on a graph or an *interval* in which the graph is increasing. In trigonometry, students confuse $\sin^{-1}x$, the inverse sine function, with $\csc x$, the reciprocal of the sine function, because the same symbol, a raised –1, is used for two different ideas. In this instance, students may benefit from recognizing that both uses involve the concept of inverse: either the multiplicative inverse or the functional inverse. In general, students need time to distinguish among uses of the same symbol with different meanings.

If we want students to be fluent with symbols, they need to be introduced to the variety of ways in which we symbolize an idea, as indicated in row 2. However, when symbols are new, we need to be careful to use just one or two initial symbolizations. As additional symbols are introduced, we must make clear their translation. We also need to help students appreciate why alternative symbolizations may be preferred in different situations. For example, in calculus, $f'(x)$ notation is helpful in writing the chain rule, but dy/dx is more helpful for implicit differentiation.

As indicated in row 3, two symbols with the same form may hide different implicit symbols. For instance, in both 3 1/2 and $3x$, symbols are next to each other. But in 3 1/2, the implicit operation is addition; in $3x$, the implicit operation is multiplication. Only the context makes the distinction clear. Here again, when evaluating $3x$ for some value of x, the implicit multiplication must be made explicit. Otherwise, students may evaluate $3x$ when $x = 4$ as 34 or may rewrite $3x$ when $x = -2$ as 3 – 2. Students need to recognize these hidden concepts, some of which are essential in problem solving.

Row 5 shows the contextual dependence of symbols that mathematicians have adopted for specific uses. Students need to be oriented to this inside knowledge. They should also be shown that although different variables are commonly used for the Pythagorean theorem, the formula for a circle centered at the origin, and the argument of a complex number, the reason that the formulas are the same is that they really represent precisely the same relationship.

Common difficulties in writing symbols

Verbalizing and reading symbols are receptive processes; producing symbols is a generative process. Students must be able to read mathematical symbols to decode other people's ideas; they must also be able to produce and transform mathematical symbols when solving problems. At this point, the previously encountered difficulties compound.

TABLE 2

Challenges in Verbalizing Symbols

Challenge	Examples
1. More than one word may be needed to verbalize a symbol or set of symbols.	$\leq$: "is less than or equal to" $\pm$: "plus or minus" $\perp$: "is perpendicular to" $C(5, 3)$: "the combination of five things taken three at a time"
2. An expression may be verbalized in multiple ways.	$x - y$: "x minus y," "x take away y," "x subtract y," "the difference between x and y," "x less y," or "y less than x" $a \div b$: "a divided by b," "the quotient of a and b," "b divides into a," or "b divides a" x^2: "x squared," "x raised to the second power," or "x to the power of 2"
3. Symbols are not always read from left to right.	$\sum_{i=1}^{n} i$ $\frac{3x^5 + 7x^3 - 8}{x^4 - 16}$
4. Inappropriate verbalizations create misunderstanding.	$-x$ read as "negative x" leads students to think that the expression must evaluate to a negative value. r^2 or r_2 read as "r-two" suggests multiplication rather than the intended squaring or subscripting. 0.6 read as "point 6" rather than "six-tenths" masks the place-value meaning of the number.

Table 4 lists difficulties that students face when they produce their own mathematical symbols. In particular, as shown in row 1, when they enter expressions into calculators or computers, they encounter a new level of translation—from two-dimensional paper to a one-dimensional line of type. Technology requires care in using variables and in introducing parentheses for grouping that may be only implicit on paper. Calculators or computers with "pretty print" are a tool that helps students verify whether they have used correct syntax in entering an expression.

Row 6 highlights the importance of using different forms of the same expression or sentence, depending on the context and the information desired from the sentence. The versatility here is similar to the use of multiple symbols for the same concept, as highlighted in **table 3.** Unless students develop this facility with symbols and the ability to recognize the desired use in a particular context, they will not be fluent users of mathematics.

Discuss: Which symbols cause the most confusion for students? What approaches have you used to remove or lessen that confusion?

Students confuse $\sin^{-1}x$ with $\csc x$ because the same symbol is used for two ideas

TABLE 3

Challenges in Reading and Understanding Symbols

Challenge	Examples
1. The same symbol may have different meanings.	A small dash may mean *opposite, minus,* or *negative.* Parentheses, as in $(-2, 3)$, can represent an ordered pair or an open interval on the number line. The raised -1 symbol can mean an inverse function or a reciprocal. The prime symbol may mean the complement of a set, feet, minutes, or the derivative.
2. Multiple symbols may represent the same concept.	Division: $12 \div 3$, $3\overline{)12}$, $\frac{12}{3}$ Multiplication: 3×4, $(3)(4)$, $3 * 4$, $3 \cdot 4$ Differentiation: dy/dx, y', or $f'(x)$ Combinations: $C(n, r)$, ${}_nC_r$, or $\binom{n}{r}$
3. Symbols may be implicit but central to understanding.	$3\ 1/2 = 3 + 1/2$. $3x = 3 \cdot x$. $x = 1x$; $x = x + 0$; $x = x^1$.
4. The placement or ordering of symbols may affect the meaning.	$34 \neq 43$ in arithmetic, but $xy = yx$ in algebra. $xy = yx$ in algebra, but xy and yx may represent different variable names in a computer-algebra system. 5.3 and 5•3 are distinct. -3^4 means the opposite of the fourth power of 3, which equals -81, whereas $(-3)^4$ means the fourth power of negative 3, which equals 81. $t_n = 2n$, $t_n = 2^n$, and $t_n = n^2$ are three major sequences whose symbolization differs only in the placement of the symbols 2 and n.
5. Specific variables may be used in specific contexts.	The same formula with different letters is used in different contexts: $a^2 + b^2 = c^2$ Pythagorean theorem $x^2 + y^2 = r^2$ Formula for a circle centered at the origin $x^2 + y^2 = z^2$ Argument of a complex number In introductory algebra, x, y, and z are typically used as unknowns; m is used for the slope of a line; and b is used for the y-intercept. In statistics, a is the y-intercept of a regression line and b is the slope. In trigonometry, θ, α, and β are typically used for angles. In algebra and calculus, f and g are common names of functions.
6. The category of a function is embedded in its symbolization. Students need to recognize different forms.	$y = mx + b$, $y - y_1 = m(x - x_1)$, and $x/a + y/b = 1$ are all forms for linear functions. $y = ax^2 + bx + c$, $y = a(x - h)^2 + k$, and $y = (x - r_1)(x - r_2)$ are all forms for quadratic functions. $y = ab^x$ and $y = ae^{kt}$ are forms for exponential functions. $y = \frac{1}{x-h} + k$ and $y = \frac{p(x)}{q(x)}$ are forms for rational functions, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$.

INSTRUCTIONAL STRATEGIES TO HELP STUDENTS READ AND USE MATHEMATICAL SYMBOLS

In general, teachers must be aware of the difficulties that symbolism creates for students. Symbolism is a form of mathematical language that is compact, abstract, specific, and formal. Unlike common English language, which students use daily, mathematical symbolism is largely limited to the mathematics classroom. Therefore, opportunities to use that language should be regular, rich, meaningful, and rewarding.

Our first precept is Bruner's guidance (1960) that learning should proceed from concrete to abstract. Mathematical symbolism and mathematical understanding are intertwined, but meaning must generally precede symbolization. Students should be involved in contexts, problems, and activities that move them from familiar to newer mathematical ideas; this stage is called the *enactive stage.* In the *iconic stage,* the products from these activities may then be expressed in tables or pictures. Ultimately, learning is expressed in common oral English with mathematical vocabulary and, in written English with mathematical symbols; this stage is called the *symbolic stage.*

We must also consider motivation. Students need to understand the value of being fluent with mathematical symbols. In some classrooms, students manage to "get by" by simply writing symbols without verbalizing them, let alone attaching meaning to them. When we ask them to do more, some are

Students need to understand the value of being fluent with mathematical symbols

resistant. Having students consider other contexts in which discussion supports an activity is helpful. Such activities include sports, where language is used to review actions or to coach people. When students begin to see the benefits of expressing mathematics in multiple ways, they become more comfortable in doing so.

Since students think and learn in a variety of ways, the following teaching ideas reflect a spectrum of approaches that use language, visualization, and projects. Many of these ideas are similar to strategies for enhancing students' fluency with mathematical vocabulary (Thompson and Rubenstein 2000).

Language strategies

Language is a familiar medium that can serve as a basis for building symbol sense. A general strategy to use when symbols are first introduced is to *say and write* the notation, emphasize relevant issues of placement or order, and give students a chance to read and record the symbols. Students may also be instructed to record symbols in their own *personal symbol table* or *card file,* in which they write the symbol, record in English how to say it, and give examples of its use.

Students' use of mathematical symbolism, however, must go deeper than saying and transcribing. We want them to own the language and use it comfortably. One method is to integrate oral work with reading mathematics. Siegel et al. (1996) offer a strategy called *say something,* in which partners read mathematics exposition and stop intermittently to share aloud emerging understandings, comments, and questions. Such work is particularly helpful in making students realize that when they read mathematics independently, they must vocalize and interpret the symbols.

Another strategy is the *silent teacher.* One of the authors stumbled on this idea when she had laryngitis. Students had to read for themselves overhead transparencies on properties of logarithms. The teacher learned that students struggled with notation that should have been familiar. Some read $\log_2 8$ as "log of two to the eighth." This discovery was an eye-opener for the author, who decided that she had too often read symbols herself for students. The silent teacher listened better to students and was able to coach them into correctly vocalizing the symbols.

Discuss: Do you too often read mathematical symbols for students? How can you modify this practice?

Writing is another powerful language strategy for supporting students' fluency with symbols. To this end, we suggest several *journal-writing* starters that focus students on the interplay between symbols and meanings and often build from the previously discussed common difficulties. The following are some examples:

1. Write three ways to verbalize $(x + 4)^3$.
2. *a*) Evaluate by hand:
   i. $\sqrt{25 + 144}$
   ii. $\dfrac{12 + 16}{24 + 26}$

   b) Evaluate (i) and (ii) with a graphing calculator. What keystrokes did you need? What symbols does the calculator need that do not appear in the written form?
3. What is the meaning of x^{-1}? What is the meaning of $f^{-1}(x)$? How is the –1 used differently in these two examples?
4. *a*) Factor $x^2 - x + 6$.

   b) Solve $x^2 - x + 6 = 0$.

   c) What are the differences between 4(*a*) and

TABLE 4

Common Difficulties in Writing Symbols

Difficulty	Examples
1. Students use incorrect syntax when translating into technology.	Implicit groupings must be made explicit: $\sqrt{9+16}$ must be entered as $\sqrt{}(9 + 16)$; $\dfrac{x^2-y^2}{x-y}$ must be entered as $(x^2 - y^2) \div (x - y)$. The negation sign may be used only with a single input; the minus sign goes between two numbers to be subtracted. The order of symbols may differ from the order used on paper, for example, the division $3\overline{)12}$ is entered as $12 \div 3$.
2. Students oversymbolize.	For twenty-five cents, students sometimes write 0.25¢. For exponentiation, students sometimes write $5^{\wedge 3}$. Students sometimes write equals signs between equations.
3. Students produce run-on sentences.	In evaluating 12(2) + 3, students write 12(2) = 24 + 3 = 27.
4. Students distribute when they should not.	In general, for a function *f*, $f(a + b) \neq f(a) + f(b)$, even though for a variable *k*, $k(x + y) = kx + ky$. Common errors include writing $(x + y)^2 = x^2 + y^2$ and log (10 + 100) = log 10 + log 100.
5. Students fail to distinguish uppercase letters from lowercase letters.	In isolating *b* in the formula for the area of a trapezoid, $A = (1/2)h(a + b)$, students must distinguish *A* from *a*.
6. Students do not recognize the value of different forms of the same expression or sentence.	The linear form $y = mx + b$ simplifies finding the initial value, rate of change, and graph. The linear form of $y - y_1 = m(x - x_1)$ is easy to produce when a slope and point are known. The quadratic form $y = a(x - h)^2 + k$ simplifies finding a vertex, $y = (x - r_1)(x - r_2)$ simplifies finding the roots, and $y = ax^2 + bx + c$ simplifies finding the *y*-intercept.

4(b)? Which one is a sentence that requires a solution?

5. Complete each analogy, and explain your thinking.
   a) Area of a square : x^2 :: Volume of a cube : ____.
   b) $a - b : a + (-b) :: a \div b$: ____.
6. Sue wrote $(x + 5)^2 = x^2 + 25$. Is Sue correct? Explain your reasoning.
7. How much money does each of the following represent? Explain.
   a) \$0.25
   b) \$25
   c) 0.25¢
8. Consider three forms for a quadratic function, and tell one advantage of each:
   a) $y = ax^2 + bx + c$
   b) $y = a(x - h)^2 + k$
   c) $y = (x - r_1)(x - r_2)$

Partner transcriptions also help students verbalize symbols. For example, one partner reads orally a symbolic expression or sentence from a textbook while the other writes in symbols what he or she hears. Then they compare the original, the spoken, and the transcribed versions. Was the oral reading accurate? Was the transcription accurate? The students can then reverse roles, and increasingly harder expressions can be used.

Silent teacher is one strategy; split-page problems is another

Often a difficulty in learning mathematical symbolism is that students record the symbols used in class, but the words that give meaning to those symbols are not recorded. Consequently, students miss the essential sense-making links. One strategy is to have students regularly do *split-page problems.* They simply fold a sheet of paper lengthwise and show the mathematical symbolism of their work on the left and the English explanation on the right. Students often report initially that the English rendition is more difficult than the mathematics. After doing these assignments regularly, they become more fluent with both the symbols and their meanings.

FACTOR!AL

e^{x}ponent

√oot

ang∠e

∫um

∆ifference

Fig. 1
Sample mathematics graffiti

Visual strategies

Student-invented *graffiti* is an enjoyable way to help students visually associate mathematical symbols with their meanings. **Figure 1** shows samples that the teacher can share with students before asking them to invent their own.

Drawing examples and nonexamples (Toumasis 1995) is another visual learning strategy that helps students, particularly in geometry. Students can be asked to illustrate examples similar to those listed here.

1. Draw a true and a false example of each statement. Label them as true or false.
   a) $AB \perp CD$.
   b) $AB \parallel CD$.
   c) $AB \cong CD$.
2. From the given diagram, write symbolic statements that convey the information shown.

Invented symbolism is another strategy that students may enjoy. As an example, one author's high school geometry teacher symbolized *bisects* with a slashed 2 ("cut in two"). He then symbolized *perpendicular bisector* with a perpendicular symbol superimposed on a 2. See **figure 2.** The author did not realize that these symbols were not conventional notation until many years later. New mathematical symbols can be invented where notation does not already exist. For example, how can we symbolize in a diagram the fact that two lengths are proportional? The invention of symbolism by students highlights the fact that people invented the symbols that we use and that students themselves may become inventors in the future. Indeed, with the growth of computers, icons and other symbol systems are being invented daily.

(a)
Bisect
(cut in two)

(b)
Perpendicular
bisector

Fig. 2
Invented symbols for bisect and perpendicular bisector

Discuss: Do you use any symbols that either you or your students have invented? How have you shared these symbols with colleagues?

Projects

Projects are another way of building students' appreciation and understanding of mathematical notation. Students can research the history of symbol systems in numeration, algebra, geometry, statistics, or other areas. For instance, modern exponentiation notation came into use only in the late 1600s (Baumgart 1989; Cajori 1993). Students can learn about symbols or procedures that differ by country even today. In another type of project, students examine popular media and collect and analyze examples that show appropriate or inappropriate uses of symbols.

SUMMARY

Using conventional mathematical symbol systems is a basic goal of mathematics curricula. However, students often have many difficulties with this language. Helping students gain fluency with symbol systems needs specific attention. We hope that educators will use and extend the ideas in this article to make the language of mathematical symbols a meaningful and accessible communication medium for mathematics learning and problem solving.

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