Egyptian Fractions

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Egyptian fractions are interesting, but seem to be pretty much useless. Strangely, they were used well into the middle ages, in Europe. In this article, I will express them using our Hindu-Arabic numerals.

2/3 or any unit fraction (fraction with one as the numerator, like 1/7) were expressed in a simple, straightforward way. 1/2 had a sign of its own (), as did 2/3 (). And the other unit fractions were just the *r*symbol (meaning "part"), with the denominator expressed as an integer, under this symbol. I drew the "*r*" after the symbol, to show that it is pronounced "r". 1/7 would be that symbol with seven, small vertical lines under it ().

Other fractions are not as simple. They are expressed as the sum of progressively smaller unit fractions. For this purpose, 2/3 is considered to be a unit fraction. For example, 2/5=1/3 + 1/15. Notice that 2/5 could also be expressed as 1/5 + 1/5. This is illegal. No two unit fractions can be the same.

Here are several more examples (for the meaning of the second column, see the addendum, below):

first calculation	**improvement**	
2/9=1/5 + 1/45	=1/6+1/18	
2/7=1/4 + 1/28		
3/8=1/4 + 1/8		
2/5=1/3 + 1/15		
3/7=1/3 + 1/11 + 1/231	=1/4+1/7+1/28	
4/9=1/3 + 1/9		
5/9=1/2 + 1/18		
4/7=1/2 + 1/14		
3/5=1/2 + 1/10		
5/8=1/2 + 1/8		
5/7=2/3 + 1/21	=1/2 + 1/7 + 1/14	
3/4=2/3 + 1/12	=1/2 + 1/4	
7/9=2/3 + 1/9		
4/5=2/3 + 1/8 + 1/120	=1/2 + 1/5 + 1/10	
5/6=2/3 + 1/6	=1/2+1/3	
6/7=2/3 + 1/6 + 1/42	=2/3 + 1/7 + 1/21	
7/8=2/3 + 1/5 + 1/120	=2/3 + 1/8 + 1/12	=1/2+1/4+1/8
8/9=2/3 + 1/5 + 1/45	=1/2+1/3+1/18	

As you can see, some of these are more complicated than others.

How did I come up with these values? Well, I estimated the fraction with the largest unit fraction that was just smaller than the given fraction. I subtracted this unit fraction from the given fraction. If this remainder was still not a unit fraction, I repeated the process, choosing the largest unit fraction that is smaller than this remainder. And the process could be repeated over and over.

Let's use 7/8 as an example. We estimate 7/8 with 2/3 (the largest unit fraction less than 7/8). We subtract 7/8 - 2/3, which is 5/24, which cannot be simplified into a unit fraction. So we estimate 5/24 with 1/5 (the largest unit fraction less than 5/24). We subtract 5/24-1/5, and we get 1/120, which is a unit fraction. So, 7/8=2/3 + 1/5 + 1/120.

This process always converges. In other words, it never goes on forever. I have proved this, but it seems that Fibonacci proved it before I did, about 1200 A.D. But this process does not guarantee the simplest Egyptian fraction. 7/8 may be expressible in two terms, instead of three. The only general method for finding the simplest Egyptian fractions, which I have found, is trial and error.

As I said at the beginning of this article, these fractions seem to be pretty much useless. For 2/9, 1/9 + 1/9 is simpler than 1/5 + 1/45. But, there may be a hidden use for these Egyptian fractions. 1/5 is a useful estimate of 2/9 (and we can throw out the tiny 1/45). The error here is only about .02 (10% of our original fraction). Dropping the third term, of some of those more complicated Egyptian fractions, gives us even smaller error. Some such estimates are fairly poor. But, the Egyptian fraction would give some indication of how big the error would be. Also, Egyptian fractions were stimulating mental exercise, which was another practical use.

And it is easier to deduce that 5/8 is greater than 3/5, as their Egyptian fractions are 5/8=1/2 + 1/8 and 3/5=1/2 + 1/10. Comparing the size of two fractions is something that we do with decimal notation. 5/8=.625 and 3/5=.6. Without decimal notation, it is very difficult to compare the relative sizes of fractions. Without decimal notation or Egyptian fractions, a person would probably have to measure out grain or sand to make a rough estimate.

An Egyptian did not say to himself, "How can I convert 5/8 to an Egyptian fraction?" There was no direct way of expressing the fraction 5/8. Instead, he was saying to himself, "What do I get when I divide 5 by 8?" We would answer 5/8. He would answer 1/2 + 1/8. And, he may have gotten his answer from a table, instead of calculating it.

Addendum:

I have corrected two mistakes in the table, above. And I have added a second column. The second column was calculated by using previous entries of the table. For example, 3/7=2/7 + 1/7. I just looked up 2/7 in the table, getting 1/4 + 1/28. I added 1/7, getting 1/4 + 1/7 + 1/28. This turns out to be better (esthetically) than 1/3 + 1/11 + 1/231, the fraction already in the table. Another example is 4/5, which is 3/5 + 1/5. I just had to look up 3/5. It is hard to tell which fraction is better, for 5/7.

It is common to write Egyptian fractions without the unit numerator and the plus sign (3/7=/4, /7, /28), approximating the way the Egyptians wrote it. I have not done that in my article above.

The proof, that the method that I described always converges in a finite number of terms, goes something like this. It is simple to show that 2/n (a reduced fraction) always has two terms (for example, 2/9=1/5 + 1/45). A little algebra is all it takes. A little more work shows that 3/n always has three or fewer terms. The argument which involves 2/n and 3/n can be generalized to show that m/n always has m or fewer terms. The same applies for the method that I used for the second column.

Evaluation (by me)

It seems that everyone is missing the point of why Egyptians used fractions like this. It seems so unnatural. The problem stems from representing Egyptian fractions using OUR notation. Written that way, it seems unproductive to have to make up a bunch of unit fraction additions to name simple fractions. What everyone seems to fail to point out is that Egyptians only had limited names for fractions. Using THOSE names, this is one solution for writing more complex fractions.

Egyptians fractions names were basically named by the size of the denominator. (see above) and in the most used cases that had specific names for particular fractions.

Imagine the difficulty of doing math if our fraction names were limited to half, third, fourth, fifth etc. There is no simple way to calculate a simpler name for half + third – it would just have to stay that way.

Our number system is limited in similar ways. What is – it would just have to stay that way.

Our number system is limited in similar ways. We have no simpler way of writing $\sqrt{3}+\sqrt{5}$!