Problem Solving Strategy Essay # 10:

Go to Extremes

James Tanton, PhD, Mathematics, Princeton 1994; MAA Mathematician in Residence

***

Teachers and schools can benefit from the chance to challenge students with interesting mathematical questions that are aligned with curriculum standards at all levels of difficulty.

This is the tenth and final broad essay in the MAA AMC Curriculum Inspirations series. (Next comes a whole slew of short pieces and examples, all pulled from the extensive MAA AMC archives, all offering moments of curriculum spark and delight for the classroom. The fun is without end!)

And our tenth featured problem-solving strategy is:

Go to Extremes

Ask something absurd!

It is fun to be quirky and to entertain ideas that have been pushed to the edge or to an extreme. It helps one develop an intuitive feel for the problem or concept at hand, to understand its limitations and restraints, and to identify the issues that seem to be at play.

Before exploring an MAA AMC query, let's illustrate this technique by seeing how four extreme questions can illuminate subtle features of the concept of an average.

1. *Can everyone have an above-average IQ?*

(This brings home the idea that an average cannot be outside the range of the set!)

2. *What is the average color of a square on a checkerboard? Gray?*

(This shows that an average need not actually be in the set!)

3. *There are nearly 7 billion people on Earth. There are eight planets in the solar system. So, on average, each planet of the solar system has 875 million humans living on it!*

(True. But this shows that an average need not be meaningful notion in all contexts!)

4. *I read somewhere that mathematicians have proved that, on average, a number has $\pi^2 / 6 \approx 1.645$ square number factors. What can that possibly mean?*

The number 1 has one square number factor: namely, 1. The number 18, for example, has two square number factors: 1 and 9. The number 1800 has eight square number factors: 1, 4, 9, 25, 36, 100, 225, and 900.

Every number has at least one square number factor (namely 1), but can have more. And the count varies from number to number.

But there are infinitely many numbers. So what can it mean to compute the "average" of an infinite list of answers?

COOL COMPUTER PROJECT: Write a program that counts the number of square factors each of the numbers from 1 to 100 possesses, and then computes the average of those 100 counts.

Extend your program to count the number of square factors for each of the numbers from 1 to 1,000. What is the average answer?

Then from 1 to 10,000. Then 1 to 100,000.

What do you think mathematicians mean by the "average" of an infinite list of answers?

OUR CHALLENGE TODAY

Let's now move on to our featured problem for today. Our query is **question 11** from the **2011 MAA AMC 10b** exam, the tenth-grade exam:

There are 52 people in a room. What is the largest value of n such that the statement "At least n people in this room have birthdays falling in the same month" is always true?

The first and a key step in the problem-solving process is:

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

This particular query induces, for me, a feeling of panic: the wording of the second sentence seems contorted and hard to unravel. It feels scary.

So … I am taking my deep breath. And I am reading through the question a second (and a third) time.

STEP 2: Understand the question. Understand the different components of the question.

I actually don't feel like I understand this question! That second sentence is just plain contorted, and telling me to "understand the question" is not helpful.

Let me calm down a bit. I can at least say something tiny about the contorted sentence: It is about people having birthdays in the same month of the year. (There, a miniscule step towards "understanding the question.")

STEP 2 CONTINUED: To understand the question, try pushing it to an extreme. Try an absurd answer and see what that tells you.

This is the featured strategy of the essay and it can help us with this second step of problem-solving.

All I "understand" right now is that the question is about people having birthdays falling in the same month of the year.

Okay … Here is something absurd:

Can a million people in this question have a birthday in the same one month?

Well, obviously not. There are only 52 people in the room!

Alright, what is a more appropriate extreme idea for this problem?

Can a 52 people in this question have a birthday in the same one month?

Yes! It is possible that everyone could have a birthday in August, for example.

So is that it? Is that the answer to the problem?

Reread the question.

In doing so the final words "always true" leap out at me. Is it <u>always true</u> that 52 people in a room have a birthday in the same month? Clearly not! This question wants something that is absolutely guaranteed.

Can we keep coming down in the extreme numbers we play with?

Can a 30 people in this question have a birthday in the same one month?

Ooh! This is the wrong question. It is true that 30 people could have the same birthday month. But we want something that is always true!

Must there always be 30 people in the room with a birthday in the same month?

No! We could have 20 people born in January, 20 born in February and 12 people born in March, for instance.

Let's keep coming down:

Must there always be 10 people in the room with a birthday in the same month?

No! We could have 9 people born in each of the first five months of the year, January through May (that's 45 people in all), and 7 people born in June.

How low should we go?

This is starting to feel tedious!

`Avoid tedious work if you can.`

Okay. So what is another extreme we could look at?

We started with the extreme of "clumping people together" all in one month. This suggests another extreme:

"Spread people out" over all the months.

How spread out can they be?

Well there are 12 months, and we can "put" one person in each month, and that accounts for 12 people. Next we can put a second person in each month (we have now accounted for 24 people). A third person in each month (36 people). A fourth (48 people), leaving 4 people to spread out over four months.

The most "spread out" extreme is to have 5 people with birthdays in each of four months, January to April say, and 4 people with birthdays in each of the remaining eight months. (And that is indeed $4\times5+8\times4=52$ people.)

What does this extreme tell us?

Well … that the numbers 4 and 5 seem significant.

Must there always be 4 people in the room with a birthday in the same month?

Well we can't have 3 people or less with the same birthday in each month (that accounts for only $12\times3=36$ people at most), so there must be at least one month with 4 people "in" it!

Must there always be 5 people in the room with a birthday in the same month?

Well, we can't have 4 people or less with the same birthday in each month, as that will account for only $12\times4=48$ people at most. So yes, there must be a month with at least 5 people "in" it.

Must there always be 6 people in the room with a birthday in the same month?

No! We already have an example to show this need not be the case: 5 people with birthdays in each of four months January to April, and 4 people with birthdays in each of the remaining eight months.

So "5" seems to be a key number. We can guarantee that, no matter how people's birthdays fall, there are at least are five people with a common birthday month. We cannot guarantee that there are 6 people with a common birthday month. (It can happen. It is just not guaranteed.)

`What was the question? Reread the question.`

I feel like that second sentence is now making sense!

What is the largest value of n such that the statement "At least n people in this room have birthdays falling in the same month" is always true?

The answer is $n=5$!

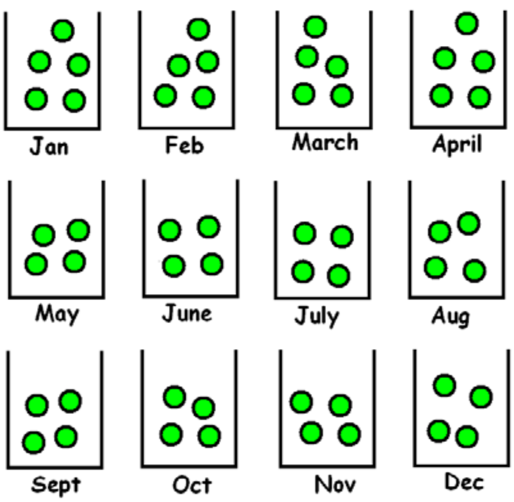
DECONSTRUCTING THE PROBLEM

Even though we have the answer, I don't yet feel we've gotten to the heart of what is going on.

To this end … Have you noticed something curious about our language? We talked about people being "put in" a month, as though we were putting people in containers labeled January, February, and so on. This invokes a visual.

And true to ESSAY 4, drawing a picture could very well help us understand our problem.

Here is a picture of our most "spread out" distribution of birthdays. In this picture, people are balls placed into containers labeled by the months.



It is now easy to <u>see</u> that we cannot have four or less balls (people) in each bin (month), as that would account for only 48 people and we have 52. Whatever we do at least one bin must have five or more balls in it.

Let's try changing the numbers to see if we can generalize matters:

<u>13 balls and 12 bins:</u> No matter how we distribute the balls at least one bin will have two or more balls in it. (Draw a picture!)

<u>29 balls and 12 bins</u>: At least one bin will have three or more balls in it. (Draw a picture – at least in your mind. Can each bin have less than three balls in it?)

<u>100 balls and 12 bins</u>: At least one bin will have 9 or more balls in it. (Can they each have 8 or less?)

Let's be extreme! <u>360,001 balls and 12 bins</u>: At least one bin will have 30,001 balls in it. (Can they each have 30,000 or less balls?)

If N balls are distributed among k bins, then at least one bin has $\frac{N}{k}$ or more balls in it.

<u>Reason</u>: What is mathematically wrong if this is not true? If each of the k bins has less than $\frac{N}{k}$ balls in it, then there are less than $k \times \frac{N}{k} = N$ balls in all. Ooops! So to have N balls in all, there must be at least one bin with at least $\frac{N}{k}$ balls in it. □

With 52 balls for 12 bins, there must be at least one bin with $\frac{52}{12} = 4\frac{1}{3}$ or more balls in it. Since we are not considering fractional balls, there must be a bin with at least 5 or more balls in it.

Students are now well poised to invent their own clever variations of this problem.

There are 820 students at our school. True or False: There are at least three students with the same birthday day?

There are 32 students in our class. True or False: There must be at least two students with the same third-to-last letter of their last names? (Test it out!)

20 students each rolled a die. True or False: At least four students, for sure, rolled the same number?

We can even be wonderfully quirky!

There are two non-bald men in New York city with exactly the same number of hairs on their heads!

<u>Reason</u>: There are about 120,000 hairs on a full head of hair. There are certainly well over a million non-bald men in New York. They can't all have different counts of hairs!

SAYING IT ANOTHER WAY: BACK TO AVERAGES

Suppose 52 balls are distributed amongst 12 bins. If we add up the count of balls in each bin we must get … 52 balls. As there are 12 bins, the average number of balls per bin is sure to be $\frac{52}{12} = 4\frac{1}{3}$. (This is true no matter how the balls happen to be distributed! Wow!)

More generally, if N balls are distributed among k bins, the average number of balls per bin is N/k.

Like human IQ, not <u>all</u> the bins can have an above average count. In the same way, not <u>all</u> the bins can have a below average count.

This leads to a rephrasing of our result:

> *Suppose N balls are distributed among k bins and not all the bins have an average count of balls.*
>
> *Then there is sure to be at least one bin with an above average count of balls, and at least one bin with a below average count of balls.*

With 52 balls and 12 bins, no bin can have the average count of balls ($4\frac{1}{3}$). So, no matter how the balls are distributed, at least one bin must have an above average count of balls (5 or more) and at least one a below average count (4 or less).

HISTORICAL COMMENT:

German mathematician Peter Gustav Lejeune Dirichlet (1805-1859) was the first to give this mathematical principle a name. He called it the *Schubfachprinzip* (drawer principle). Today it is known either as the pigeonhole principle, as Dirichlet's principle, or as the cubby-hole principle. He realized it can lead to powerful, if not profound, results. (Think non-bald men in New York!)

The simplest version of the principle reads:

> **The Pigeonhole Principle**: If more than k objects are to go into k bins, then at least one bin must contain more than one object.

For example, if 13 pigeons are to lodge into 12 cubbies, then at least one cubby must contain two or more pigeons. <u>Note</u>: This does not mean that exactly one cubby will be sure to contain exactly two pigeons! The principle says instead, that for each and every possible distribution of pigeons, there is always sure to be at least one cubby with two or more pigeons in it.

The version of the principle we've developed is called the …

> **Generalized Pigeonhole Principle**: If N objects are to go into k bins, then at least one bin must contain N/k or more objects.

Question: In a state lottery one is required to submit a four-digit number composed of non-zero digits. (For example, 7823 and 8828 are valid entries, but 8906 is not.) If 10,000 people enter the lottery, how many people, for certain, submitted the same number?

To explore the pigeon-hole principle and its curious applications further, see www.jamestanton.com/?p=1315 .

CURRICULUM CONNECTIONS:

The pigeon-hole principle does not, in and of itself, make an appearance in the standard curriculum. But I would argue that our MAA AMC exploration has accomplished two feats:

1. We have deepened our understanding of "average" or *mean* as a descriptive measure of central tendency. (A passive use of "mean.")

and

2. We have shown how the mean can be used in modeling real-world phenomena. (A surprising active use of "mean.")

COMMON CORE STATE STANDARDS and PRACTICES:

From the high-school curriculum we have touched on part of the Common Core State Standard from the statistics strand:

S-ID: 2 *Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets.*

We have also linked with the grades 6-8 statistics strands with each mention of "mean."

In looking at non-bald men in New York City we have demonstrated an, admittedly short but superbly elegant, application of Mathematical Practices standard:

MP4: *Model with mathematics*

(What other biological or sociological facts about the real world can you deduce from the pigeon-hole principle?)

Also, through problem solving, and possibly conducting this essay as a class discussion/activity, we have modeled the practice standards:

MP1: *Make sense of problems and persevere in solving them.*
MP2: *Reason abstractly and quantitatively.*
MP3: *Construct viable arguments and critique the reasoning of others.*
MP7: *Look for and make use of structure.*

SOME FINAL EXAMPLES:

A "passive use" question of mean:

a) Give an example of five data values that have mean 10 and median 1000.
b) Give an example of five data values that have median 10 and mean 1000.

Repeat for six data values.

(If your curriculum discusses "mode" try finding six data values with mean 10, mode 10 and median 1000.)

Some "active use" questions of mean:

1. How many people need to be in a room to ensure that at least two people have the same pair of initials? (For example, my name is "James Tanton" and so my initials are "JT.") At least seven people with the same pair of initials?

2. Give an interesting value for N for which you can be sure the statement "There are at least N Americans with the same number of hairs on their heads" is true.

3. Are there two novels with exactly the same number of words in them?

Curriculum Inspirations is brought to you by the Mathematical Association of America and the MAA American Mathematics Competitions.