Algebra Toolkit

1 *Rules of Thumb.*

- Make sure that you can prove all formulas you use. This is even better than memorizing the formulas. Although it is best to memorize, as well.
- Strive for elegant, economical methods. Look for symmetry. Look for invariants.
- Sometimes symmetry and economy conflict. For example, it is not always best to square both sides of equations like $\sqrt{A}+\sqrt{B}=C$. Sometimes it makes better sense to change it to $\sqrt{A}=C-\sqrt{B}$, and then square. (Example: 2006B/15).
- Zero is your best friend. One is your second-best friend.
- Never multiply out, unless you have to. Always look for factorizations, instead.
- To know the zeros of a polynomial is to know the polynomial.
- Look for telescoping terms.
- Don't worry about being clever. Dumb methods work, too. Low-tech is better than high-tech. But if things start getting too dirty and messy, step back and ask yourself if there is a better way to proceed.

2 *Arithmetic.*

- Know all squares up to $\lceil\sqrt{2008}\rceil^2 = 45^2 = 2025$
- Know all perfect powers < 2008
- Know all factorials up to 10!
- Know the first 9 or 10 rows of Pascal's Triangle
- Factor the current year!
- Know your primes, at least under 100, ideally up to 200 or so.
- Be able to mentally square numbers and multiply numbers, using the factorization $x^2-y^2=(x-y)(x+y)$. For example, to compute 73^2, use $73^2-3^2=70*76=4900+420=5320$.

3 *Factoring.*

- $1001 = 7\times 11\times 13$
- $(x+y)^2 = x^2+2xy+y^2$.
- $(x-y)^2 = x^2-2xy+y^2$.
- $(x+y)^3 = x^3+3x^2y+3xy^2+y^3 = x^3+y^3+3xy(x+y)$.
- $(x-y)^3 = x^3-3x^2y+3xy^2-y^3 = x^3-y^3-3xy(x-y)$.
- $x^2-y^2=(x-y)(x+y)$.
- $x^n-y^n=(x-y)(x^{n-1}+x^{n-2}y+x^{n-3}y^2+\cdots+y^{n-1})$ for all n.
- $x^n+y^n=(x+y)(x^{n-1}-x^{n-2}y+x^{n-3}y^2-\cdots+y^{n-1})$ for all odd n (the terms of the second factor alternate in sign).
- $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ac)$

4 *Polynomials.*

- FTA, conjugate complex solutions
- Completing the square
- Factor/Remainder theorem
- Relationship between roots and coefficients

5 *Sequences and Series.*

- Arithmetic series: The best sum formula is $S = n(first + last)/2$, since it tends to get you thinking about averages which gets you thinking about symmetry.
- Know the formulas for sums of squares and cubes. Be able to derive formulas for sums of higher powers if needed.
- TELESCOPING is the mother of all sequence/summation methods. Know the classic telescopes, for example $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ and $(n+1)! - n! = n \cdot n!$. And don't forget that telescoping can be used with products as well. You can always turn products into sums with logarithms.
- You don't really need to learn the "calculus of differences" to handle recurrence sequence problems. Almost always, low-tech methods (usually involving telescoping!) suffice.
- Even though it is really a calculus topic, you should be able to prove that the harmonic series $1 + 1/2 + 1/3 + 1/4 + \cdots$ diverges (without calculus). Likewise, you should be able to prove that $1 + 1/4 + 1/9 + 1/16 + \cdots$ converges, again, without calculus.
- The sum of the first n odd numbers is equal to n^2. This is a remarkably fruitful fact.

6 *Miscellaneous.*

- Floor and ceiling functions
- Floor functions and counting multiples; lattice points.
- Absolute value
- Logarithms: Be able to prove the all-important, and easy-to-memorize formula $\log_a b \log_b c = \log_a c$.
- Complex numbers: Cis form, sums of roots of unity, take absolute value of both sides.

Number Theory Toolkit

1 Primes and Divisibility

FTA (Fundamental Theorem of *Arithmetic*) Every positive integer can be factored *uniquely* into primes. This factorization is often called the PPF (prime-power factorization).

GCD and LCM Greatest common divisor and least common multiple, respectively. If $\text{GCD}(a,b) = 1$ we say that a and b are **relatively prime** and denote this by $a \perp b$.

- If $g|a$ and $g|b$ then $g|(a+b), g|(a-b)$, etc. In fact, $g|(ax+by)$ for any integers x, y. The expression $ax+by$ is called a **linear combination** of a and b.
- Consequently, every pair of consecutive positive integers is relatively prime. Same is true of consecutive odd integers. Consecutive even integers have a GCD of 2.
- *Linear combination rule*: The GCD of a and b is a linear combination of a and b. In fact, it is smallest positive linear combination of a and b. Thus if $a \perp b$, there exist integers x, y such that $ax + by = 1$.
- The coefficients x, y above can be found by performing the *Euclidean algorithm* backwards. In practice, they can be found by inspection for small numbers. For example, $5 \perp 7$ and $3 \cdot 5 + (-2) \cdot 7 = 1$. (Don't worry if you don't know the Euclidean algorithm.)

Number of divisors is denoted by $\tau(n)$ and includes 1 and n in the count. So $\tau(p) = 2$ for any prime p and in general, $\tau(p^a q^b \cdots) = (a+1)(b+1)\cdots$ when the number is in PPF form. (Some books use $d(n)$ instead of $\tau(n)$ but Greek letters are so much more sophisticated.)

Sum of divisors is denoted by $\sigma(n)$ and includes 1 and n. The formula is rather simple. Make sure you can see why it's true (try simple examples such as $n = 12$ and $n = 300$):

$$\sigma(p^a q^b \cdots) = (1 + p + p^2 + \cdots + p^a)(1 + q + q^2 + \cdots + q^b) \cdots.$$

2 Modular Arithmetic

Congruence notation The notation $x \equiv y \pmod{m}$ means the following equivalent things. All of them are worth internalizing.

- $x - y$ is a multiple of m
- x has the remainder y when divided by m, provided that $0 \le y < m$.
- $x = mK + y$ for some integer K

Thus $x \equiv 3 \pmod{4}$ means that x has a remainder of 3 when divided by 4, and it also means that you can write x "in the form" $4k+3$. It is common to use small negative numbers in congruences, especially -1. For example, $99 \equiv -1 \pmod{4}$. We could have written $99 \equiv 3 \pmod{4}$, but the former is just as true, and often more useful.

Congruence algebra Congruence notation is incredibly useful because you can add, subtract, and multiply (*but not divide*) both just as you can with ordinary equality. For example,

- If $a \equiv b \pmod{m}$ and $x \equiv y \pmod{m}$, then $a + x \equiv b + y \pmod{m}$ and $ax \equiv by \pmod{m}$

- $100000 \equiv 10 \pmod{11}$, since $10 \equiv -1 \pmod{11}$ and thus $10^5 \equiv (-1)^5 \pmod{11}$.

Divisibility Rules You should know the divisibility rules for 2, 3, 4, 5, 6, 8, 9, and 11, using congruences or other methods.

Mod n analysis is a crucial start of any problem. Put it into mod 2, 3, and 4 filters, sometimes more. For example, you should verify that all perfect squares are congruent to 0 or 1 (mod 4) only. (A mod 2 analysis is also called a parity analysis.)

Fermat's Little Theorem states that if p is a prime, and a is not a multiple of p, then

$$a^{p-1} \equiv 1 \pmod{p}.$$

For example, $4^{12} \equiv 1 \pmod{13}$.

3 Diophantine Equations

- Use of parity, mod 3, mod 4 analysis
- Linear equations
- $x^2 - y^2 = n$. The AIME uses this humble equation in endless ways. Become intimate with it.
- Pythagorean equation
- $x^2 + y^2 = n$

Mostly Number Theory

1 *Classics.*

(a) Find the smallest integer greater than 1 which has a remainder of 1 upon division by $2, 3, 4, 56, 7, 8, 9, 10$. Find the smallest positive integer which has a remainder of $1, 2, 3, \ldots, 9$ when divided by $2, 3, \ldots, 10$, respectively.

(b) Lockers in a row are numbered $1, 2, 3, \ldots, 1000$. At first, all the lockers are closed. A person walks by and opens every other locker, starting with locker #2. Thus lockers $2, 4, 6, \ldots, 998, 1000$ are open. Another person walks by, and changes the "state" (i.e., closes a locker if it is open, opens a locker if it is closed) of every third locker, starting with locker #3. Then another person changes the state of every fourth locker, starting with #4, etc. This process continues until no more lockers can be altered. Which lockers will be closed?

(c) Find all integer solutions to $x^2 + y^2 + z^2 = 2xyz$.

(d) Does the function $f(x) = x^2 + x + 41$ always output primes for each positive integer x?

(e) Find all primitive Pythagorean triples, i.e., primitive solutions to $x^2 + y^2 = z^2$.

(f) Investigate divisibility patterns of the Fibonacci numbers.

(g) Now that you have investigated parity (mod 2 values) of the elements of Pascal's Triangle, it is time to investigate the mod p values!

(h) Is it possible for four consecutive integers to be composite? How about five? More than five? Arbitrarily many?

(i) Show that, for any natural number k, the product of k consecutive numbers is divisible by $k!$

2 Show that if $a^2+b^2=c^2$, then $3|ab$.

3 If $x^3+y^3=z^3$, show that one of the three must be a multiple of 7.

4 Make sure that you know why 100! ends in 24 zeros and 1000! ends in 249 zeros. Can $n!$ end with $n/4$ zeros?

5 Find the smallest positive integer n such that $\tau(n)=10$.

6 Find the remainder when 2^{1000} is divided by 13. (This was an AHMSE problem when I was in high school.)

7 Let P be the product of the first 100 positive odd integers. Find the largest integer k such that P is divisible by 3^k.

8 Prove that among any 12 consecutive positive integers there is at least one which is smaller than the sum of its proper divisors. (The proper divisors of a positive integer n are all positive integers other than 1 and n which divide n. For example, the proper divisors of 14 are 2 and 7.)

9 Prove that any integer greater than or equal to 7 can be written as a sum of two relatively prime integers, both greater than 1. (Two integers are relatively prime if they share no common positive divisor other than 1. For example, 22 and 15 are relatively prime, and thus $37=22+15$ represents the number 37 in the desired way.)

10 What kind of numbers can be written as the sum of two or more consecutive integers? For example, 10 is such a number, because $10=1+2+3+4$. Likewise, $13=6+7$ also works.

11 Each of the following are products of two primes. Only one of these products can be written as the sum of the cubes of two positive integers. Which one?

[A] 104729×8512481779 [B] 104729×8242254443 [C] 104761×8242254443
[D] 104761×11401596337 [E] 104729×11401596337

12 A point whose coordinates are both integers is called a lattice point. How many lattice points lie on the hyperbola $x^2-y^2=2000^2$?

13 What is the smallest positive integer with six positive odd integer divisors and twelve positive even divisors?

14 How many ordered pairs (x,y) of integers are solutions to

$$\frac{xy}{x+y}=99?$$

15 The following two numbers shown are both primes written in standard base-10 notation.

$$6786x681, \quad 6786x683.$$

Notice that all the digits of the numbers are the same, except for the final digit, but we are not telling you what the 5th digit is. If you take the product of these two numbers, and then find the sum of the digits of the result, and then find the sum of the digits of that, and so on, until you get a single digit, what do you get?

16 Find x^2+y^2 if $x,y \in \mathbf{N}$ and

$$xy+x+y=71, \quad x^2y+xy^2=880.$$

17 The numbers in the sequence

$$101, 104, 109, 116, \ldots$$

are of the form $a_n = 100+n^2$, where $n = 1,2,3,\ldots$. For each n, let d_n be the greatest common divisor of a_n and a_{n+1}. Find the maximum value of d_n as n ranges through the positive integers.

18 How many positive integer multiples of 1001 can be expressed in the form $10^j - 10^i$, where i, j are integers and $0 \le i < j \le 99$?

19 Two positive integers differ by 60. The sum of their square roots is the square root of an integer that is not a perfect square. What is the maximum possible sum of the two integers?

20 Let $f(n)$ denote the sum of the digits of n.

(a) For any integer n, prove that eventually the sequence

$$f(n), f(f(n)), f(f(f(n))), \ldots$$

will become constant. This constant value is called the **digital sum** of n.

(b) Prove that the digital sum of the product of any two twin primes, other than 3 and 5, is 8. (Twin primes are primes that are consecutive odd numbers, such as 17 and 19.)

(c) (IMO 1975) Let $N = 4444^{4444}$. Find $f(f(f(n)))$, without a calculator.

21 Find the last three digits of 7^{9999}.

22 For a deck containing an even number of cards, define a "perfect shuffle" as follows: divide the deck into two equal halves, the top half and the bottom half; then interleave the cards one by one between the two halves, starting with the top card of the bottom half, then the top card of the top half, etc. For example, if the deck has 6 cards, labeled "123456" from top to bottom, after a perfect shuffle the order of the cards will be "415263." Determine the minimum (positive) number of perfect shuffles needed to restore a 94-card deck to its original order. Can you generalize this to decks of arbitrary (even) size?

23 Let $\{a_n\}_{n\geq 0}$ be a sequence of integers satisfying $a_{n+1} = 2a_n + 1$. Is there an a_0 so that the sequence consists entirely of prime numbers?

24 Find all non-negative integral solutions $(n_1, n_2, \ldots, n_{14})$ to

$$n_1^4 + n_2^4 + \cdots + n_{14}^4 = 1{,}599.$$

25 For positive integers let $\tau(n)$ denote the number of positive integer divisors of n including 1 and n. Define $S(n) = \tau(1) + \tau(2) + \cdots + \tau(n)$. Let a denote the number of positive integers $n \leq 2005$ with $S(n)$ odd, and let b denote the number of positive integers $n \leq 2005$ with $S(n)$ even. Find $|a - b|$.

26 For a certain integer k there are exactly 70 positive integers $n_1, n_2, \ldots, n_{70}$ such that

$$k = \lfloor \sqrt[3]{n_1} \rfloor = \lfloor \sqrt[3]{n_2} \rfloor = \cdots = \lfloor \sqrt[3]{n_{70}} \rfloor$$

and k divides n_i for all i such that $1 \leq i \leq 70$.

Find the maximum value of n_i/k for $1 \leq i \leq 70$.

27 Let S be the set of integers between 1 and 2^{40} whose binary expansions have exactly two 1's. If a number is chosen at random from S, the probability that it is divisible by 9 is m/n where m and n are relatively prime positive integers. Find $m + n$.

28 A triangular array of squares has one square in the first row, two in the second, and in general, k squares in the kth row for $1 \leq k \leq 11$. With the exception of the bottom row, each square rests on two squares in the row immediately below (illustrated in diagram). In each square of the eleventh row, a 0 or a 1 is placed. Numbers are then placed into the other squares, with the entry for each square being the sum of the entries in the two squares below it. For how many initial distributions of 0's and 1's in the bottom row is the number in the top square a multiple of 3?

29 Find the largest integer satisfying the following conditions:

(i) n^2 can be expressed as the difference of two consecutive cubes; (ii) $2n + 79$ is a perfect square.

Miscellaneous Optional Topics

1 *The Euclidean* [illegible]. Repeated use of the division algorithm allows one to easily compute the GCD of two numbers. For example, we shall compute $(333,51)$:

$$\begin{aligned} 333 &= 6\cdot 51+27; \\ 51 &= 1\cdot 27+24; \\ 27 &= 1\cdot 24+3; \\ 24 &= 8\cdot 3+0. \end{aligned}$$

We start by dividing 333 by 51. Then we divide 51 by the remainder from the previous step. At each successive step, we divide the last remainder by the previous remainder. We do this until the remainder is zero, and our answer—the GCD—is the final non-zero remainder (in this case, 3).

Here is another example. To compute $(89,24)$, we have

$$\begin{aligned} 89 &= 3\cdot 24+17; \\ 24 &= 1\cdot 17+7; \\ 17 &= 2\cdot 7+3; \\ 7 &= 2\cdot 3+1; \end{aligned}$$

so the GCD is 1.

This method is called the **Euclidean Algorithm.** Explain why it works!

2 *Linear Diophantine Equations.* Since $17 \perp 11$, there exist integers x, y such that $17x+11y=1$. For example, $x=2, y=-3$ work. Here is a neat trick for generating more integer solutions to $17x+11y=1$: Just let

$$x=2+11t, \quad y=-3-17t,$$

where t is *any* integer.

(a) Verify that $x=2+11t, y=-3-17t$ will be a solution to $17x+11y=1$, no matter what t is. This is a simple algebra exercise, and is really just a nice example of the add zero creatively tool.

(b) Show that *all* integer solutions to $17x+11y=1$ have this form; i.e., if x and y are integers satisfying $17x+11y=1$, then $x=2+11t, y=-3-17t$ for some integer t.

(c) It was easy to find the solution $x=2, y=-3$ by trial and error, but for larger numbers we can use the Euclidean algorithm *in reverse*. For example, use the example in Problem 1 to find x, y such that $89x+24y=1$. Start by writing 1 as a linear combination of 3 and 7; then write 3 as a linear combination of 7 and 17; etc.

(d) Certainly if $x=2, y=-3$ is a solution to $17x+11y=1$, then $x=2u, y=-3u$ is a solution to $17x+11y=u$. And as above, verify that *all* solutions are of the form $x=2u+11t, y=-3u-17t$.

(e) This method can certainly be generalized to any linear equation of the form $ax+by=c$, where a, b, c are constants. First we divide both sides by the GCD of a and b; if this GCD is *not* a divisor of c there cannot be solutions. Then we find a single solution either by trial and error, or by using the Euclidean algorithm.

(f) To see another example of generating infinitely many solutions to a diophantine equation, look at the problems about Pell's Equation below.

3 *The Chinese Remainder Theorem.* Consider the following simultaneous congruence.

$$\begin{aligned} x &\equiv 3 \pmod{11}, \\ x &\equiv 5 \pmod{6}. \end{aligned}$$

It is easy to find a solution, $x = 47$, by inspection. Here's another method. Since $6 \perp 11$, we can find a linear combination of 6 and 11 that equals one, for example, $(-1) \cdot 11 + 2 \cdot 6 = 1$. Now compute

$$5 \cdot (-1) \cdot 11 + 3 \cdot 2 \cdot 6 = -19.$$

This number is a solution, modulo $66 = 6 \cdot 11$. Indeed, $47 \equiv -19 \pmod{66}$.

(a) Why does this work?

(b) Note that the two moduli (which were 11 and 6 in the example) must be relatively prime. Show by example that there may not always be a solution to a simultaneous congruence if the two moduli share a factor.

(c) Let $m \perp n$, let a and b be arbitrary, and let x simultaneously satisfy the congruences $x \equiv a \pmod{m}$ and $x \equiv b \pmod{n}$. The algorithm described above will produce a solution for x. Show that this solution is *unique* modulo mn.

(d) Show that this algorithm can be extended to any finite number of simultaneous congruences, as long as the moduli are pairwise relatively prime.

(e) Show that there exist three consecutive numbers, each of which is divisible by the 1999th power of an integer.

(f) Show that there exist 1999 consecutive numbers, each of which is divisible by the cube of an integer.

4 *Pell's Equation.* The quadratic diophantine equation $x^2 - dy^2 = n$, where d and n are fixed, is called **Pell's equation**. We will mostly restrict our attention to the cases where $n = \pm 1$. For a fuller treatment of this subject, including the relationship between Pell's equation and continued fractions, consult just about any number theory textbook.

(a) Notice that if d is negative, then $x^2 - dy^2 = n$ has only finitely many solutions.

(b) Likewise, if d is perfect square, then $x^2 - dy^2 = n$ has only finitely many solutions.

(c) Consequently, the only "interesting" case is when d is positive and not a perfect square. Let us consider a concrete example: $x^2 - 2y^2 = 1$.

1. It is easy to see by inspection that $(1,0)$ and $(3,2)$ are solutions. A bit more work yields the next solution: $(17,12)$.
2. Cover the next line so you can't read it! Now, see if you can find a simple linear recurrence that produces $(3,2)$ from $(1,0)$ and produces $(17,12)$ from $(3,2)$. Use this to produce a new solution, and check to see if it works.

3. You discovered that if (u,v) is a solution to $x^2-2y^2=1$, then so is $(3u+4v,2u+3v)$. Prove why this works. It is much easy to see why it works than to discover it in the first place, so don't feel bad if you "cheated" above.
4. But now that you understand the lovely tool of **generating new solutions** from clever linear combinations of old solutions, you should try your hand at $x^2-8y^2=1$. In general, this method will furnish infinitely many solutions to Pell's equation for any positive non-square d.

(d) Notice that $(3+2\sqrt{2})^2=17+12\sqrt{2}$. Is this a coincidence? Ponder, conjecture, generalize.

(e) Try to find solutions to $x^2-dy^2=-1$, for a few positive non-square values of d.

(f) An integer is called square-full if each of its prime factors occurs to at least the second power. Prove that there exist infinitely many pairs of consecutive square-full integers.

Supplemental Problems for Monday

1 *Quickies.* Try to do these as quickly as possible, without calculator, of course.

(a) A perfect power is a number of the form a^b, where a and b are positive integers. Find two perfect powers that differ by 100. Did you find more than one solution? Can you find solutions that don't involve perfect squares?

(b) Find two perfect powers that differ by 10.

(c) Find all integer solutions to $x^2+x+1=y^2$.

(d) Solve the system of equations

$$\begin{aligned} 2x_1+x_2+x_3+x_4+x_5 &= 6 \\ x_1+2x_2+x_3+x_4+x_5 &= 12 \\ x_1+x_2+2x_3+x_4+x_5 &= 18 \\ x_1+x_2+x_3+2x_4+x_5 &= 24 \\ x_1+x_2+x_3+x_4+2x_5 &= 30. \end{aligned}$$

(e) What is the first time after 12 o'clock that the hour and minute hands meet? This is an amusing and moderately hard algebra exercise, well worth doing if you never did it before. However, this problem can be solved in a few seconds *in your head* if you avoid messy algebra and just consider the "natural" point of view. Go for it!

2 Find all integers n such that 10^n+1 is prime.

3 *The classic "Chicken Nuggets" problem.* Bay Area Rapid Food sells vegan soy chicken-flavored nuggets in two sizes: boxes with 7 or 11 nuggets. Clearly, one can purchase 18 nuggets (one box of each size) but it is impossible to purchase exactly 17 nuggets. What is the largest number of nuggets that *cannot* be purchased by ordering boxes (and not wasting nuggets)? Generalize and investigate.

4 Find, the nearest percent, the fraction of elements of the first googol rows of Pascal's Triangle that are even.

5 Let a and b be integers greater than one which have no common divisors. Prove that

$$\sum_{i=1}^{b-1}\left\lfloor \frac{ai}{b}\right\rfloor = \sum_{j=1}^{a-1}\left\lfloor \frac{bj}{a}\right\rfloor,$$

and find the value of this common sum.

6 For positive integers n, define S_n to be the minimum value of the sum

$$\sum_{k=1}^{n}\sqrt{(2k-1)^2+a_k^2},$$

as the $a_1,a_2,\ldots,a_n$ range through all positive values such that

$$a_1+a_2+\cdots+a_n=17.$$

Find S_{10}.

Mostly Algebra

1 Let C be the coefficient of x^2 in the expansion of the product

$$(1-x)(1+2x)(1-3x)\cdots(1+14x)(1-15x).$$

Find $|C|$.

2 Find the remainder when

$$100^2+99^2-98^2-97^2+96^2+\cdots+4^2+3^2-2^2-1^2$$

is divided by 1000.

3 For which integer n is $1/n$ closest to $\sqrt{1{,}000{,}000}-\sqrt{999{,}999}$?

4 Let $x=\sqrt[3]{1000}-\sqrt[3]{999}$. What integer is closest to $1/x$?

5 Solve $x^4+x^3+x^2+x+1=0$.

6 Let

$$x=\frac{4}{(\sqrt{5}+1)(\sqrt[4]{5}+1)(\sqrt[8]{5}+1)(\sqrt[16]{5}+1)}.$$

Find $(x+1)^{48}$.

7 A virus is placed into a colony of 1,998 bacteria. Every minute, each virus destroys one bacterium apiece, after which all the bacteria and viruses divide in two. For example, after one minute, there will be $1997\times 2=3994$ bacteria and 2 viruses. After two minutes, there will be 3992×2 bacteria and 4 viruses, etc. How long (in minutes) will it take for the all the bacteria to be destroyed?

8 Factor z^5+z+1.

9 Solve $z^6+z^4+z^3+z^2+1=0$.

10 The polynomial

$$P(x)=(1+x+x^2+\cdots+x^{17})^2-x^{17}$$

has 34 complex roots of the form $z_k=r_k[\cos(2\pi a_k)+i\sin(2\pi a_k)], k=1,2,3,\ldots,34$, with $0<a_1\le a_2\le a_3\le\cdots\le a_{34}<1$ and $r_k>0$. Given that $a_1+a_2+a_3+a_4+a_5=m/n$, where and are relatively prime positive integers, find $m+n$.

11 If $P(x)$ denotes a polynomial of degree n such that $P(k)=k/(k+1)$ for $k=0,1,2,\ldots,n$, determine $P(n+1)$.

12 Prove that

$$\left\lfloor\frac{n+2^0}{2^1}\right\rfloor+\left\lfloor\frac{n+2^1}{2^2}\right\rfloor+\left\lfloor\frac{n+2^2}{2^3}\right\rfloor+\cdots+\left\lfloor\frac{n+2^{n-1}}{2^n}\right\rfloor=n$$

for any positive integer n.

13 Find the minimum value of $xy+yz+xz$, given that x,y,z are real and $x^2+y^2+z^2=1$. No calculus, please!

14 Find all integer solutions (n,m) to

$$n^4+2n^3+2n^2+2n+1=m^2.$$

15 If $x^2+y^2+z^2=49$ and $x+y+z=x^3+y^3+z^3=7$, find xyz.

16 Let r,s,t be the roots of $8x^3+1001x+2008=0$. Find $(r+s)^3+(s+t)^3+(t+r)^3$.

17 Find all real values of x that satisfy $(16x^2-9)^3+(9x^2-16)^3=(25x^2-25)^3$.

18 Given that is a complex number such that $z+\frac{1}{z}=2\cos 3°$, find the least integer that is greater than $z^{2000}+\frac{1}{z^{2000}}$.

19 (*Crux Mathematicorum*, June/July 1978) Show that n^4-20n^2+4 is composite when n is any integer.

20 There exist unique positive integers x and y that satisfy the equation $x^2+84x+2008=y^2$. Find $x+y$.

21 *2008 II:6*. The sequence $\{a_n\}$ is defined by

$$a_0=1, a_1=1, \quad \text{and } a_n=a_{n-1}+\frac{a_{n-1}^2}{a_{n-2}} \text{ for } n\geq 2.$$

The sequence $\{b_n\}$ is defined by

$$b_0=1, b_1=3, \quad \text{and } b_n=b_{n-1}+\frac{b_{n-1}^2}{b_{n-2}} \text{ for } n\geq 2.$$

Find b_{32}/a_{32}.

22 Let m be a positive integer, and let $a_0,a_1,\ldots,a_m$ be a sequence of real numbers such that $a_0=37, a_1=72, a_m=0$, and

$$a_{k+1}=a_{k-1}-\frac{3}{a_k}$$

for $k=1,2,\ldots,m-1$. Find m.

23 Find a formula for the sum $1\cdot 1!+2\cdot 2!+3\cdot 3!+\cdots+n\cdot n!$.

24 Every positive integer has a unique factorial base expansion $(f_1,f_2,\ldots,f_m)$, meaning that

$$k=1!\cdot f_1+2!\cdot f_2+3!\cdot f_3+\cdots+m!\cdot f_m,$$

where each f_i is an integer, $0\leq f_i\leq i$, and $0<f_m$. Given that $(f_1,f_2,\ldots,f_j)$ is the factorial base expansion of

$$16!-32!+48!-64!+\cdots+1968!-1984!+2000!,$$

find the value of

$$f_1-f_2+f_3-f_4+\cdots+(-1)^{j+1}f_j.$$

25 Prove that

$$\sqrt{\frac{1}{\left(\frac{1}{1729}-\frac{22}{7}\right)^2}+\frac{1}{\left(\frac{22}{7}-\frac{355}{113}\right)^2}+\frac{1}{\left(\frac{355}{113}-\frac{1}{1729}\right)^2}}$$

is rational.

26 A sequence is defined as follows: $a_1 = a_2 = 3$, and for $n \geq 2$, $a_{n+1}a_{n-1} = a_n^2 + 2007$. Find the greatest integer that does not exceed $\frac{a_{2006}^2+a_{2007}^2}{a_{2006}a_{2007}}$.

27 (E. Johnston) Let S be the set of positive integers which do not have a zero in their base-10 representation; i.e.,

$$S = \{1,2,\ldots,9,11,12,\ldots,19,21,\ldots\}.$$

Does the sum of the reciprocals of the elements of S converge or diverge?

28 Let $A = \sum_{n=1}^{10000} \frac{1}{\sqrt{n}}$. Find $\lfloor A \rfloor$ without a calculator.

29 Given that x, y, z are real numbers that satisfy:

$$x = \sqrt{y^2 - \frac{1}{16}} + \sqrt{z^2 - \frac{1}{16}}$$

$$y = \sqrt{z^2 - \frac{1}{25}} + \sqrt{x^2 - \frac{1}{25}}$$

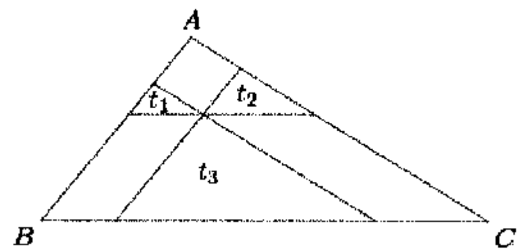
$$z = \sqrt{x^2 - \frac{1}{36}} + \sqrt{y^2 - \frac{1}{36}}$$

and that $x + y + z = m/\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime, find $m + n$.

AMC PREP

Geometry Problems for Extra Study

1. A point P is chosen in the interior of $\triangle ABC$ such that when lines are drawn through P parallel to the sides of $\triangle ABC$, the resulting smaller triangles t_1, t_2, and t_3 in the figure, have areas 4, 9, and 49, respectively. Find the area of $\triangle ABC$.



2. Circles of radii 5, 5, 8, and m/n are mutually externally tangent, where m and n are relatively prime positive integers. Find $m+n$.

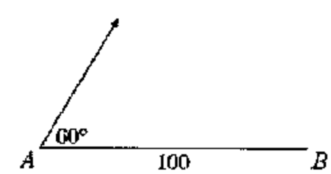
3. Consider the parallelogram with vertices (10, 45), (10, 114), (28, 153), and (28, 84). A line through the origin cuts this figure into two congruent polygons. The slope of the line is m/n, where m and n are relatively prime positive integers. Find $m+n$.

4. Let $ABCD$ be a parallelogram. Extend DA through A to a point P, and let PC meet AB at Q and DB at R. Given that $PQ = 735$ and $QR = 112$, find RC.

5. Consider the set of points that are inside or within one unit of a rectangular parallelepiped (box) that measures 3 by 4 by 5 units. Given that the volume of this set is $\frac{m+n\pi}{p}$, where m, n, and p are positive integers, and n and p are relatively prime, find $m+n+p$.

6. A wooden cube, whose edges are one centimeter long, rests on a horizontal surface. Illuminated by a point source of light that is x centimeters directly above an upper vertex, the cube casts a shadow on the horizontal surface. The area of the shadow, which does not include the area beneath the cube, is 48 square centimeters. Find the greatest integer that does not exceed $1000x$.

7. Two skaters, Allie and Billie, are at points A and B, respectively, on a flat, frozen lake. The distance between A and B is 100 meters. Allie leaves A and skates at a speed of 8 meters per second on a straight line that makes a $60°$ angle with AB. At the same time Allie leaves A, Billie leaves B at a speed of 7 meters per second and follows the straight path that produces the earliest possible meeting of the two skaters, given their speeds. How many meters does Allie skate before meeting Billie?



AMC PREP

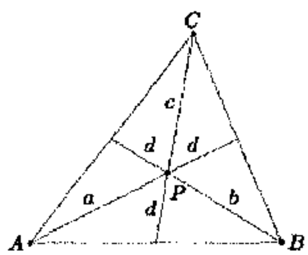
Geometry Problems for Extra Study

8. Three circles, each of radius 3, are drawn with centers at (14, 92), (17, 76), and (19, 84). A line passing through (17, 76) is such that the total area of the parts of the three circles to one side of the line is equal to the total area of the parts of the three circles to the other side of it. What is the absolute value of the slope of this line?

9. Rhombus $PQRS$ is inscribed in rectangle $ABCD$ so that vertices P, Q, R, and S are interior points on sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, respectively. It is given that $PB = 15$, $BQ = 20$, $PR = 30$, and $QS = 40$. Let m/n, in lowest terms, denote the perimeter of $ABCD$. Find $m + n$.

10. Let $ABCDE$ be a convex pentagon with $AB||CE$, $BC||AD, AC||DE$, $\angle ABC = 120°$, $AB = 3$, $BC = 5$, and $DE = 15$. Given that the ratio between the area of triangle ABC and the area of triangle EBD is m/n, where m and n are relatively prime positive integers, find $m + n$. (2004, #13)

11. Let P be an interior point of triangle ABC and extend lines from the vertices through P to the opposite sides. Let a, b, c, and d denote the lengths of the segments indicated in the figure. Find the product abc if $a + b + c = 43$ and $d = 3$.



12. A right circular cone has a base with radius 600 and height $200\sqrt{7}$. A fly starts at a point on the surface of the cone whose distance from the vertex of the cone is 125, and crawls along the surface of the cone to a point on the exact opposite side of the cone whose distance from the vertex is $375\sqrt{2}$. Find the least distance that the fly could have crawled.

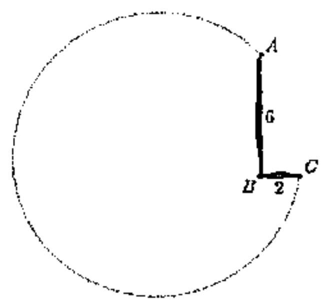
13. Let w_1 and w_2 denote the circles $x^2+y^2+10x-24y-87=0$ and $x^2+y^2-10x-24y+153=0$, respectively. Let m be the smallest positive value of a for which the line $y = ax$ contains the center of a circle that is internally tangent to w_1 and externally tangent to w_2. Given that $m^2 = \frac{p}{q}$, where p and q are relatively prime positive integers, find $p + q$.

AMC PREP

Geometry Problems for Class Discussion

1. In $\triangle ABC$, $AB = 13$, $BC = 15$, and $CA = 14$. Point D is on $\overline{BC}$ with $CD = 6$. Point E is on $\overline{BC}$ such that $\angle BAE \cong \angle CAD$. Given that $BE = \frac{p}{q}$, where p and q are relatively prime positive integers, find q.

2. A machine-shop cutting tool has the shape of a notched circle, as shown. The radius of the circle is $\sqrt{50}$ cm, the length of AB is 6 cm, and that of BC is 2 cm. The angle ABC is a right angle. Find the square of the distance (in centimeters) from B to the center of the circle.

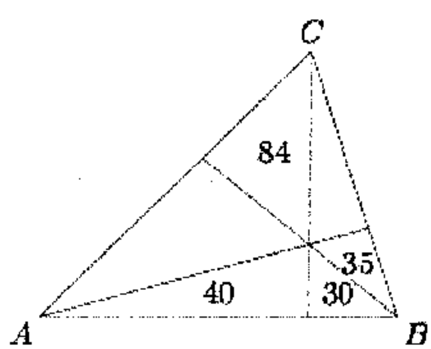


3. In $\triangle ABC$, $AB = 425$, $BC = 450$, and $AC = 510$. An interior point P is then drawn, and segments are drawn through P parallel to the sides of the triangle. If these three segments are of an equal length d, find d.

4. $ABCD$ is a rectangular sheet of paper that has been folded so that corner B is matched with point B' on edge AD. The crease is EF, where E is on AB and F is on CD. The dimensions $AE = 8$, $BE = 17$, and $CF = 3$ are given. The perimeter of rectangle $ABCD$ is m/n, where m and n are relatively prime positive integers. Find $m + n$.

5. Prove the Angle-Angle Similarity Theorem.

6. As shown in the figure, triangle ABC is divided into six smaller triangles by lines drawn from the vertices through a common interior point. The areas of four of these triangles are as indicated. Find the area of triangle ABC.



AMC PREP

Geometry Problems for Class Discussion

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7. Prove the Power of a Point Theorem. Specifically, if chords AB and CD of $\odot O$ meet at X inside $\odot O$, the prove that $(AX)(XB) = (CX)(XD)$.

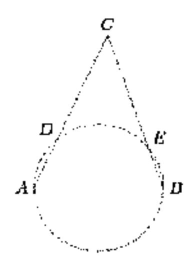
8. $ABCD$ is a cyclic quadrilateral (the quadrilateral can be inscribed in a circle) with $AB = 6$. $BC = 9$ and $CD = 8$. Diagonals AC and BD meet at M, such that $BM = 2DM$. Find AD.

9. A square has sides of length 2. Set S is the set of all line segments that have length 2 and whose endpoints are on adjacent sides of the square. The midpoints of the line segments in set S enclose a region whose area to the nearest hundredth is k. Find $100k$.

10. Faces ABC and BCD of tetrahedron $ABCD$ meet at an angle of $30°$. The area of face ABC is 120, the area of face BCD is 80, and $BC = 10$. Find the volume of the tetrahedron.

11. Circles C_1 and C_2 are externally tangent, and they are both internally tangent to circle C_3. The radii of C_1 and C_2 are 4 and 10, respectively, and the centers of the three circles are collinear. A chord of C_3 is also a common external tangent of C_1 and C_2. Given that the length of the chord is $\frac{m\sqrt{n}}{p}$, where m, n, and p are positive integers, m and p are relatively prime, and n is not divisible by the square of any prime, find $m + n + p$.

12. In the figure, ABC is a triangle and $AB = 30$ is a diameter of the circle. If $AD = AC/3$ and $BE = BC/4$, then what is the area of the triangle?



13. Triangle ABC has right angle at B, and contains a point P for which $PA = 10$, $PB = 6$, and $\angle APB = \angle BPC = \angle CPA$. Find PC.

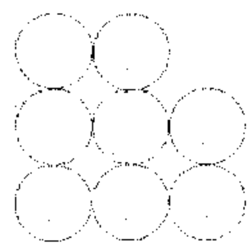
14. A car travels due east at 2/3 miles per minute on a long, straight road. At the same time, a circular storm, whose radius is 51 miles, moves southeast at $\sqrt{2}/2$ mile per minute. A time $t = 0$, the center of the storm is 110 miles due north of the car. At time $t = t_1$ minutes, the car enters the storm circle, and at time $t = t_2$ minutes, the car leaves the storm circle. Find $(t_1 + t_2)/2$.

15. Trapezoid $ABCD$ has sides $AB = 92$, $BC = 50$, $CD = 19$, and $AD = 70$, with AB parallel to CD. A circle with center P on AB is drawn tangent to BC and AD. Given that $AP = \frac{m}{n}$, where m and n are relatively prime positive integers, find $m + n$.

AMC PREP

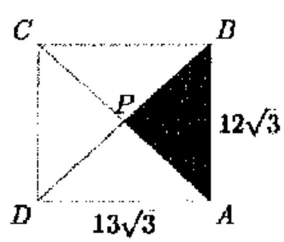
Geometry Problems for Class Discussion

16. Eight circles of diameter 1 are packed in the first quadrant of the coordinate plane as shown. Let region $\mathcal{R}$ be the union of the eight circular regions. Line ℓ, with slope 3, divides $\mathcal{R}$ into two regions of equal area. Line ℓ's equation can be expressed in the form $ax = by + c$, where a, b, and c are positive integers whose greatest common divisor is 1. Find $a^2 + b^2 + c^2$.



17. Three of the edges of a cube are AB, BC, and CD, and AD is an interior diagonal. Points P, Q, and R are on AB, BC, and CD, respectively, so that $AP = 5, PB = 15, BQ = 15$, and $CR = 10$. What is the area of the polygon that is the intersection of plane PQR and the cube?

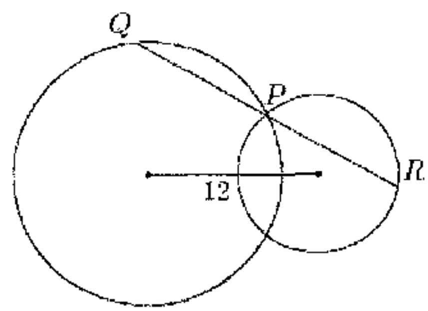
18. The rectangle $ABCD$ below has dimensions $AB = 12\sqrt{3}$ and $BC = 13\sqrt{3}$. Diagonals $\overline{AC}$ and $\overline{BD}$ intersect at P. If triangle ABP is cut out and removed, edges $\overline{AP}$ and $\overline{BP}$ are joined, and the figure is then creased along segments $\overline{CP}$ and $\overline{DP}$, we obtain a triangular pyramid, all four of whose faces are isosceles triangles. Find the volume of this pyramid.



19. A beam of light strikes BC at point C with angle of inclination $x = 19.94°$ and reflects with an equal angle of reflection as shown. The light beam continues its path, reflecting off line segments AB and BC according to the rule: angle of incidence equals angle of reflection. Given that $y = \frac{x}{10} = 1.994°$ and $AB = BC$, determine the number of times the light beam will bounce off the two line segments. Include the first reflection at C in your count.

20. In a certain circle, the chord of a d-degree arc is 22 centimeters long, and the chord of a $2d$-degree arc is 20 centimeters longer than the chord of a $3d$-degree arc, where $d < 120$. The length of the chord of a $3d$-degree arc is $-m + \sqrt{n}$ centimeters, where m and n are positive integers. Find $m + n$.

21. In the adjoining figure, two circles with radii 6 and 8 are drawn with their centers 12 units apart. At P, one of the points of intersection, a line is drawn in sich a way that the chords QP and PR have equal length. (P is the midpoint of QR) Find the square of the length of QP.



AMC PREP

Counting Problems for Class Discussion

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1. Let S be a set with six elements. Let $\mathcal{P}$ be the set of all subsets of S. Subsets A and B of S, not necessarily distinct, are chosen independently and at random from $\mathcal{P}$. The probability that B is contained in at least one of A or $S-A$ is $\frac{m}{n^r}$, where m, n, and r are positive integers, n is prime, and m and n are relatively prime. Find $m+n+r$.

2. Many states use a sequence of three letters followed by a sequence of three digits as their standard license-plate pattern. Given that each three-letter three-digit arrangement is equally likely, the probability that such a license plate will contain at least one palindrome (a three-letter arrangement or a three-digit arrangement that reads the same left-to-right as it does right-to-left) is m/n, where m and n are relatively prime positive integers. Find $m+n$.

3. Ten points are marked on a circle. How many distinct convex polygons of three or more sides can be drawn using some (or all) of the ten points as vertices?

4. The numbers 1447, 1005, and 1231 have something in common: each is a 4-digit number beginning with 1 that has exactly two identical digits. How many such numbers are there?

5. Given a rational number, write it as a fraction in lowest terms and calculate the product of the resulting numerator and denominator. For how many rational numbers between 0 and 1 will $20!$ be the resulting product?

6. When a certain biased coin is flipped five times, the probability of getting heads exactly once is not equal to 0 and is the same as that of getting heads exactly twice. Let i/j, in lowest terms, be the probability that the coin comes up heads in exactly 3 out of 5 flips. Find $i+j$.

7. Nine tiles are numbered $1, 2, 3, \ldots, 9$, respectively. Each of three players randomly selects and keeps three of the tiles, and sums those three values. The probability that all three players obtain an odd sum is m/n, where m and n are relatively prime positive integers. Find $m+n$.

8. What fraction of all permutations of the numbers 1, 2, 3, 4, 5, 6 such that the first term is not 1 has third term 3?

9. Let n be the number of ordered quadruples (x_1, x_2, x_3, x_4) of positive odd integers that satisfy $\sum_{i=1}^{4} x_i = 98$. Find $\frac{n}{100}$.

10. Two squares of a checkerboard are painted yellow, and the rest are painted green. Two color schemes are equivalent if one can be obtained from the other by applying a rotation in the plane board. How many inequivalent color schemes are possible?

11. A fair coin is to be tossed ten times. Let $\frac{i}{j}$, in lowest terms, be the probability that heads never occur on consecutive tosses. Find $i + j$.

12. Two mathematicians take a morning coffee break each day. They arrive at the cafeteria independently, at random times between 9 a.m. and 10 a.m., and stay for exactly m minutes. The probability that either one arrives while the other is in the cafeteria is 40%, and $m = a - b\sqrt{c}$, where a, b, and c are positive integers, and c is not divisible by the square of any prime. Find $a + b + c$.

13. A convex polyhedron has for its faces 12 squares, 8 regular hexagons, and 6 regular octagons. At each vertex of the polyhedron one square, one hexagon, and one octagon meet. How many segments joining vertices of the polyhedron lie in the interior of the polyhedron rather than along an edge or a face?

14. 6 sprinters are in the 100-meter dash. Ties are allowed in the final standings. There are N different possible orders of finish. Find the remainder when N is divided by 1000.

15. A $150 \times 324 \times 375$ rectangular solid is made by gluing together $1 \times 1 \times 1$ cubes. An internal diagonal of this solid passes through the interiors of how many of the $1 \times 1 \times 1$ cubes?

16. Let p be the probability that, in the process of repeatedly flipping a fair coin, one will encounter a run of 5 heads before one encounters a run of 2 tails. Given that p can be written in the form m/n where m and n are relatively prime positive integers, find $m + n$.

Supplemental Problems for Monday

1 *Quickies.* Try to do these as quickly as possible, without calculator, of course.

(a) A perfect power is a number of the form a^b, where a and b are positive integers. Find two perfect powers that differ by 100. Did you find more than one solution? Can you find solutions that don't involve perfect squares?

(b) Find two perfect powers that differ by 10.

(c) Find all integer solutions to $x^2+x+1=y^2$.

(d) Solve the system of equations

$$\begin{aligned} 2x_1+x_2+x_3+x_4+x_5 &= 6\\ x_1+2x_2+x_3+x_4+x_5 &= 12\\ x_1+x_2+2x_3+x_4+x_5 &= 18\\ x_1+x_2+x_3+2x_4+x_5 &= 24\\ x_1+x_2+x_3+x_4+2x_5 &= 30. \end{aligned}$$

(e) What is the first time after 12 o'clock that the hour and minute hands meet? This is an amusing and moderately hard algebra exercise, well worth doing if you never did it before. However, this problem can be solved in a few seconds *in your head* if you avoid messy algebra and just consider the "natural" point of view. Go for it!

2 Find all integers n such that 10^n+1 is prime.

3 *The classic "Chicken Nuggets" problem.* Bay Area Rapid Food sells vegan soy chicken-flavored nuggets in two sizes: boxes with 7 or 11 nuggets. Clearly, one can purchase 18 nuggets (one box of each size) but it is impossible to purchase exactly 17 nuggets. What is the largest number of nuggets that *cannot* be purchased by ordering boxes (and not wasting nuggets)? Generalize and investigate.

4 Find, the nearest percent, the fraction of elements of the first googol rows of Pascal's Triangle that are even.

5 Let a and b be integers greater than one which have no common divisors. Prove that

$$\sum_{i=1}^{b-1}\left\lfloor\frac{ai}{b}\right\rfloor=\sum_{j=1}^{a-1}\left\lfloor\frac{bj}{a}\right\rfloor,$$

and find the value of this common sum.

6 For positive integers n, define S_n to be the minimum value of the sum

$$\sum_{k=1}^{n}\sqrt{(2k-1)^2+a_k^2},$$

as the $a_1,a_2,\ldots,a_n$ range through all positive values such that

$$a_1+a_2+\cdots+a_n=17.$$

Find S_{10}.

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AMC PREP

Counting Problems

1. An ordered pair (m, n) of non-negative integers is called "simple" if the addition $m+n$ in base 10 requires no carrying. Find the number of simple ordered pairs of non-negative integers that sum to 1492.

2. A jar has 10 red candies and 10 blue candies. Terry picks two candies at random, then Mary picks two of the remaining candies at random. Given that the probability that they get the same color combination, irrespective of order, is m/n, where m and n are relatively prime positive integers, find $m+n$.

3. Twenty five of King Arthur's knights are seated at their customary round table. Three of them are chosen - all choices of three being equally likely - and are sent off to slay a troublesome dragon. Let P be the probability that at least two of the three had been sitting next to each other. If P is written as a fraction in lowest terms, what is the sum of the numerator and denominator?

4. Let $(a_1, a_2, a_3, \ldots, a_{12})$ be a permutation of (1, 2, 3, ..., 12) for which $a_1 > a_2 > a_3 > a_4 > a_5 > a_6$ and $a_6 < a_7 < a_8 < a_9 < a_{10} < a_{11} < a_{12}$. An example of such a permutation is (6, 5, 4, 3, 2, 1, 7, 8, 9, 10, 11, 12). Find the number of such permutations.

5. A gardener plants three maple trees, four oak trees and five birch trees in a row. He plants them in random order, each arrangement being equally likely. Let m/n in lowest terms be the probability that no two birch trees are next to one another. Find $m+n$.

6. Let S be a set with six elements. In how many different ways can one select two not necessarily distinct subsets of S so that the union of the two subsets is S? The order of selection does not matter; for example, the pair of subsets $\{a, c\}$, $\{b, c, d, e, f\}$ represents the same selection as the pair $\{b, c, d, e, f\}$, $\{a, c\}$.

7. Two notches are made randomly on a stick. The stick is then broken at those two notches, to create three pieces. What is the probability that these three pieces can form the sides of a triangle?

8. Let A, B, C, and D be the vertices of a regular tetrahedron, each of whose edges measure 1 meter. A bug, starting from vertex A, observes the following rule: at each vertex it chooses one of the three edges meeting at that vertex, each edge being equally likely to be chosen, and crawls along that edge to the vertex at its opposite end. Let $p = \frac{n}{729}$ be the probability that the bug is at vertex A when it has crawled exactly 7 meters. Find the value of n.

9. Ten points in the plane are given, with no three collinear. Four distinct segments joining pairs of these points are chosen at random, all such segments being equally likely. The probability that some three of the segments form a triangle whose vertices are among the ten given points is m/n, where m and n are relatively prime positive integers. Find $m+n$.

10. In a sequence of coin tosses, one can keep a record of instances in which a tail is immediately followed by a head, a head is immediately followed by a head, and etc. We denote these by TH, HH, and etc. For example, in the sequence TTHHTHTTTHHTTH of 15 coin tosses we observe that there are two HH, three HT, four TH, and five TT subsequences. How many different sequences of 15 coin tosses will contain exactly two HH, three HT, four TH, and five TT subsequences?

11. A collection of 8 cubes consists of one cube with edge-length k for each integer k, $1 \le k \le 8$. A tower is to be built using all 8 cubes according to the rules:

    - Any cube may be the bottom cube in the tower.
    - The cube immediately on top of a cube with edge-length k must have edge-length at most $k+2$.

    Let T be the number of different towers than can be constructed. What is the remainder when T is divided by 1000?

12. Forty teams play a tournament in which every team plays every other team exactly once. No ties occur, and each team has a 50% chance of winning any game it plays. The probability that no two teams win the same number of games is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $\log_2 n$.

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$$a_1+a_2+\cdots+a_n=17.$$

Find S_{10}.

Mostly Algebra

1 Let C be the coefficient of x^2 in the expansion of the product

$$(1-x)(1+2x)(1-3x)\cdots(1+14x)(1-15x).$$

Find $|C|$.

2 Find the remainder when

$$100^2+99^2-98^2-97^2+96^2+\cdots+4^2+3^2-2^2-1^2$$

is divided by 1000.

3 For which integer n is $1/n$ closest to $\sqrt{1{,}000{,}000}-\sqrt{999{,}999}$?

4 Let $x=\sqrt[3]{1000}-\sqrt[3]{999}$. What integer is closest to $1/x$?

5 Solve $x^4+x^3+x^2+x+1=0$.

6 Let

$$x=\frac{4}{(\sqrt{5}+1)(\sqrt[4]{5}+1)(\sqrt[8]{5}+1)(\sqrt[16]{5}+1)}.$$

Find $(x+1)^{48}$.

7 A virus is placed into a colony of 1,998 bacteria. Every minute, each virus destroys one bacterium apiece, after which all the bacteria and viruses divide in two. For example, after one minute, there will be $1997\times 2=3994$ bacteria and 2 viruses. After two minutes, there will be 3992×2 bacteria and 4 viruses, etc. How long (in minutes) will it take for the all the bacteria to be destroyed?

8 Factor z^5+z+1.

9 Solve $z^6+z^4+z^3+z^2+1=0$.

10 The polynomial

$$P(x)=(1+x+x^2+\cdots+x^{17})^2-x^{17}$$

has 34 complex roots of the form $z_k=r_k[\cos(2\pi a_k)+i\sin(2\pi a_k)], k=1,2,3,\ldots,34$, with $0<a_1\le a_2\le a_3\le\cdots\le a_{34}<1$ and $r_k>0$. Given that $a_1+a_2+a_3+a_4+a_5=m/n$, where and are relatively prime positive integers, find $m+n$.

11 If $P(x)$ denotes a polynomial of degree n such that $P(k)=k/(k+1)$ for $k=0,1,2,\ldots,n$, determine $P(n+1)$.

12 Prove that

$$\left\lfloor\frac{n+2^0}{2^1}\right\rfloor+\left\lfloor\frac{n+2^1}{2^2}\right\rfloor+\left\lfloor\frac{n+2^2}{2^3}\right\rfloor+\cdots+\left\lfloor\frac{n+2^{n-1}}{2^n}\right\rfloor=n$$

for any positive integer n.

13 Find the minimum value of $xy+yz+xz$, given that x,y,z are real and $x^2+y^2+z^2=1$. No calculus, please!

14 Find all integer solutions (n,m) to

$$n^4+2n^3+2n^2+2n+1=m^2.$$

15 If $x^2+y^2+z^2=49$ and $x+y+z=x^3+y^3+z^3=7$, find xyz.

16 Let r,s,t be the roots of $8x^3+1001x+2008=0$. Find $(r+s)^3+(s+t)^3+(t+r)^3$.

17 Find all real values of x that satisfy $(16x^2-9)^3+(9x^2-16)^3=(25x^2-25)^3$.

18 Given that is a complex number such that $z+\frac{1}{z}=2\cos 3^\circ$, find the least integer that is greater than $z^{2000}+\frac{1}{z^{2000}}$. Ans = 0

19 (*Crux Mathematicorum*, June/July 1978) Show that n^4-20n^2+4 is composite when n is any integer.

20 There exist unique positive integers x and y that satisfy the equation $x^2+84x+2008=y^2$. Find $x+y$.

21 The sequence $\{a_n\}$ is defined by

$$a_0=1, a_1=1, \quad \text{and } a_n=a_{n-1}+\frac{a_{n-1}^2}{a_{n-2}} \text{ for } n\geq 2.$$

The sequence $\{b_n\}$ is defined by

$$b_0=1, b_1=3, \quad \text{and } b_n=b_{n-1}+\frac{b_{n-1}^2}{b_{n-2}} \text{ for } n\geq 2.$$

Find b_{32}/a_{32}.

22 Let m be a positive integer, and let $a_0,a_1,\ldots,a_m$ be a sequence of real numbers such that $a_0=37, a_1=72, a_m=0$, and

$$a_{k+1}=a_{k-1}-\frac{3}{a_k}$$

for $k=1,2,\ldots,m-1$. Find m.

23 Find a formula for the sum $1\cdot 1!+2\cdot 2!+3\cdot 3!+\cdots+n\cdot n!$.

24 Every positive integer has a unique factorial base expansion $(f_1,f_2,\ldots,f_m)$, meaning that

$$k=1!\cdot f_1+2!\cdot f_2+3!\cdot f_3+\cdots+m!\cdot f_m,$$

where each f_i is an integer, $0\leq f_i\leq i$, and $0<f_m$. Given that $(f_1,f_2,\ldots,f_j)$ is the factorial base expansion of

$$16!-32!+48!-64!+\cdots+1968!-1984!+2000!,$$

find the value of

$$f_1-f_2+f_3-f_4+\cdots+(-1)^{j+1}f_j.$$

25 Prove that

$$\sqrt{\frac{1}{\left(\frac{1}{1729}-\frac{22}{7}\right)^2}+\frac{1}{\left(\frac{22}{7}-\frac{355}{113}\right)^2}+\frac{1}{\left(\frac{355}{113}-\frac{1}{1729}\right)^2}}$$

is rational.

26 A sequence is defined as follows: $a_1 = a_2 = 3$, and for $n \geq 2$ $a_{n+1}a_{n-1} = a_n^2 + 2007$. Find the greatest integer that does not exceed $\frac{a_{2006}^2+a_{2007}^2}{a_{2006}a_{2007}}$

27 (E. Johnston) Let S be the set of positive integers which do not have a zero in their base-10 representation; i.e.,

$$S = \{1,2,\ldots,9,11,12,\ldots,19,21,\ldots\}.$$

Does the sum of the reciprocals of the elements of S converge or diverge?

28 Let $A = \sum_{n=1}^{10000} \frac{1}{\sqrt{n}}$. Find $\lfloor A \rfloor$ without a calculator.

29 Given that x, y, z are real numbers that satisfy:

$$x = \sqrt{y^2 - \frac{1}{16}} + \sqrt{z^2 - \frac{1}{16}}$$

$$y = \sqrt{z^2 - \frac{1}{25}} + \sqrt{x^2 - \frac{1}{25}}$$

$$z = \sqrt{x^2 - \frac{1}{36}} + \sqrt{y^2 - \frac{1}{36}}$$

and that $x + y + z = m/\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime, find $m + n$.

AIME PREP Problems

Problems Written By Chris Jeuell (chrisje@microsoft.com)

Problem 1. The lengths of the sides of a triangle with positive area are $\log_{10} 12, \log_{10} 75$, and $\log_{10} n$, where n is a positive integer. Find the number of possible values of n.

Problem 2. For each positive integer k, let S_k denote the increasing arithmetic sequence of integers whose first term is 1 and whose common difference is k. For example, S_3 is the sequence $1, 4, 7, \ldots$. For how many values of k does S_k contain the term 2005?

Problem 3. Consider the increasing sequence $1, 2, 3, 11, 12, 13, 21, \ldots$, which consists of all positive integers, all of whose digits are $1, 2$, or 3. How many digits (not necessarily distinct) does the 2007^{th} term of this sequence have?

Problem 4. The number

$$\sqrt{104\sqrt{6} + 468\sqrt{10} + 144\sqrt{15} + 2006}$$

can be written as $a\sqrt{2} + b\sqrt{3} + c\sqrt{5}$, where a, b, and c are positive integers. Find abc.

Problem 5. Compute the number of ordered pairs (x, y) of integers with $1 \le x < y \le 100$ such that $i^x + i^y$ is a real number, where $i^2 = -1$.

Problem 6. An integer between 1000 and 9999, inclusive, is called *balanced* if the sum of its two leftmost digits equals the sum of its two rightmost digits. How many balanced integers are there?

Problem 7. Compute the sum of all integers a for which the polynomial $x^2+ax+10a$ can be factored over the integers.

Problem 8. The polynomial $P(x)$ is cubic. What is the largest value of k for which the polynomials $Q_1(x) = x^2 + (k-29)x - k$ and $Q_2(x) = 2x^2 - (2k-43)x + k$ are both factors of $P(x)$?

Problem 9. A number is called a *palindrome* if it reads the same when its digits are read in reverse order. Call a positive integer *almost palindromic* if it is <u>not</u> a palindrome, but the number that results by swapping some two of its digits <u>is</u> a palindrome. How many six-digit positive integers are almost palindromic?

Problem 10. A frog is placed at the origin on the number line, and moves according to the following rule: in a given move, the frog advances to either the closest point with a greater integer coordinate that is a multiple of 3, or to the closest point with a greater integer coordinate that is a multiple of 13. A *move sequence* is a sequence of coordinates which correspond to valid moves, beginning with 0, and ending with 39. For example, $0, 3, 6, 13, 15, 26, 39$ is a move sequence. How many move sequences are possible for the frog?

Problem 11. The sequence $a_1, a_2, \ldots$ is geometric with $a_1 = a$ and common ratio r, where a and r are positive integers. Given that $\log_8 a_1 + \log_8 a_2 + \cdots + \log_8 a_{12} = 2006$, find the number of possible ordered pairs (a, r).

Problem 12. Triangle ABC lies in the Cartesian plane and has area 70. The coordinates of B and C are $(12, 19)$ and $(23, 20)$, respectively, and the coordinates of A are (p, q). The line containing the median to side BC has slope -5. Find the largest possible value of $p + q$.

Problem 13. Triangle ABC is a right triangle with $AC = 7, BC = 24$, and right angle at C. Point M is the midpoint of $\overline{AB}$, and D is on the same side of line AB as C so that $AD = BD = 15$. Given that the area of $\triangle CDM$ can be expressed as $\frac{m\sqrt{n}}{p}$, where m, n, and p are positive integers, m and p are relatively prime, and n is not divisible by the square of any prime, find $m + n + p$.

Problem 14. There are two distinguishable flagpoles, and there are 19 flags, of which 10 are identical blue flags, and 9 are identical green flags. Let N be the number of distinguishable arrangements using all of the flags in which each flagpole has at least one flag and no two green flags on either pole are adjacent. Find the remainder when N is divided by 1000.

Problem 15. For all $x \geq 1$, consider the function $f(x)$, defined by

$$f(x) = \begin{cases} \lfloor x \rfloor \cdot \left| x - \lfloor x \rfloor - \frac{1}{2\lfloor x \rfloor} \right| & \text{if } x < \lfloor x \rfloor + \frac{1}{\lfloor x \rfloor}, \\ f\left(x - \frac{1}{\lfloor x \rfloor}\right) & \text{otherwise.} \end{cases}$$

Let $g(x) = 2^{x-2007}$. Compute the number of points in which the graphs of f and g intersect.