Calculus Individual
March Regional

Throughout this test, NOTA stands for "None of These Answers"

1. Given an odd polynomial f and an even polynomial g. $f(g'(g(f'(f(g(g(x)))))))$ is:

A) Always even. B) Always odd. C) Sometimes odd. D) Neither even nor odd. E) NOTA

2. Given two parallel lines in a plane, with points A and B on one and C on the other. $\overline{AB}=b$ and initially $\overline{AC}=h$ and $\overline{AC}\perp\overline{AB}$ where b and h are positive constants. C moves along its parallel line away from B with a constant speed v. In terms of b, h, and v, what is the rate of change of the area of ABC?

A) $\frac{b+h}{v}$ B) $\frac{vh}{2}$ C) v D) 0 E) NOTA

3. What is the volume formed by revolving the graph of $|x|+|y|=1$ around the x-axis?

A) $\frac{\pi}{3}$ B) $\frac{2\pi}{3}$ C) π D) 2π E) NOTA

4. Evaluate: $\dfrac{\int_0^{180}\sin(x^\circ)dx}{\int_0^{\pi}\sin(x)dx}$ A) 1 B) $\frac{\pi}{180}$ C) $\frac{180}{\pi}$ D) $\frac{360}{\pi}$ E) NOTA

5. What is the volume formed by revolving a circle of radius 1 around a line in the same plane as the circle passing through the center of the circle?

A) $\frac{3\pi}{4}$ B) π C) $\frac{4\pi}{3}$ D) 2π E) NOTA

6. $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{b-a}{n}\left(a+\frac{k(b-a)}{n}\right)=$

A) $\frac{a+b}{2}$ B) $\frac{a^2+b^2}{2}$ C) $\frac{ab}{2}$ D) $\frac{b^2-a^2}{2}$ E) NOTA

7. The Pour Baby Soft Drink company sells cans of soda for \$1.00 each. They are planning a sweepstakes game where, on average for a given x, 1 in every x cans has a coupon for another can free. Assuming all coupons issued will be used, let $f(x)$ give the minimum new amount that they must now charge per can in order to not lose any revenue from the sweepstakes. What is $f'(12)$?

A) $\frac{1}{12}$ B) $\frac{13}{12}$ C) $\frac{1}{144}$ D) DNE E) NOTA

8. Let P(n), for positive integers n, equal the number of digits required to number the pages of a book that is n pages long, starting with page 1. For instance, P(13) = 17 because the required digits are 1, 2, 3, 4, 5, 6, 7, 8, 9, 1, 0, 1, 1, 1, 2, 1, and 3. Evaluate: $\lim_{n\to\infty}\frac{P(n)}{n}$

A) ∞ B) 0 C) 1 D) Does not exist E) NOTA

9. Evaluate: $\int_0^2 \int_1^3 xy \cdot dx \cdot dy =$

A) 2 B) 4 C) 9 D) 16 E) NOTA

10. $\lim_{x \to \sqrt{2}} \left(\int_{-x}^{x} \sqrt{x^2 - s^2}\, ds - \left(9^2 + \frac{\left(\frac{19}{x}\right)^2}{11} \right)^{x^{-4}} \right) =$

A) 0 B) 1 C) 2 D) 3 E) NOTA

11. Given $f\left(\frac{1}{x}\right) = \frac{1}{x+1}$, what is $f'\left(\frac{1}{2}\right)$?

A) $-\frac{1}{4}$ B) $\frac{4}{9}$ C) $\frac{2}{3}$ D) $\frac{9}{4}$ E) NOTA

12. Dienda and Bofa start at the origin and simultaneously begin running (to the right) along the graphs of $y = \sin(x)$ and $y = \frac{x}{4\pi}$, respectively, and each with a constant speed of 1 unit/sec. How many times after they start are they at the same point?

A) 0 B) 1 C) 2 D) 3 E) NOTA

13. Evaluate, rounded to 5 decimal places (use trig substitution!): $\int_0^{\sqrt{2}} \frac{dx}{\sqrt{1+x^2}} =$

A) 1.14620 B) 1.14621 C) 1.14622 D) 1.14623 E) NOTA

14. What is the center of mass of the region bounded by $y = \sqrt{3} \cdot |x|$ and the line $y = 3$?

A) (0, 0) B) (0, 1) C) (0, 2) D) (0, 3) E) NOTA

15. Let $P(n)$ give the probability that in a random sample of n people, there exist two people who share the same birthday (discount leap years, etc). What is $P(2005) - P(2004)$?

A) 0 B) $\frac{1}{365}$ C) Undefined D) 1 E) NOTA

For questions 16 through 18, let **I** be the ice-cream-cone-shaped region bounded by the graphs of $y = \sqrt{1-x^2}$ and $y = 3|x| - 3$:

16. What is the area of **I**?

A) $\frac{\pi}{2}+6$ B) $\frac{\pi+6}{2}$ C) $\pi+3$ D) $\pi+6$ E) NOTA

17. What is the total volume enclosed when **I** is revolved around the y-axis?

A) $\frac{\pi}{3}$ B) $\frac{5\pi}{3}$ C) 2π D) $\frac{7\pi}{3}$ E) NOTA

18. The union of **I** and the set of points enclosed by $x^2+(y-1)^2=1$ forms a kind of "double-scoop-cone." What is its area?

A) $\frac{7\pi}{6}+3$ B) $\frac{5\pi}{6}+3+\frac{\sqrt{3}}{2}$ C) $\frac{3\pi}{2}+3$ D) $\frac{4\pi}{3}+3+\frac{\sqrt{3}}{2}$ E) NOTA

19. Let f and g be functions such that $f(x)^2 = g^{-1}(4x-1)$. What is $f(x)f'(x)g'\left(f(x)^2\right)$?

A) Not enough information B) 2 C) $\frac{2}{4x-1}$ D) $8x-2$ E) NOTA

20. Two random real numbers a and b are chosen such that $0<a<2$ and $0<b<4$. What is the probability that $a^2>b$, to the nearest hundredth?

A) .64 B) .65 C) .66 D) .67 E) NOTA

21. Pump A fills a pool at a rate of t^n gallons/minute and drain B drains the pool at a rate of t^{n+1} gallons/minute. If the pump and drain are turned on simultaneously and the pool is initially empty, at what t will it next be empty?

A) $1+\frac{1}{n-1}$ B) $1+\frac{1}{n}$ C) $1+\frac{1}{n+1}$ D) It fills forever E) NOTA

22. A particle moves according to: $x(t)=3\sin t$ and $y(t)=4\cos t$. What is the magnitude of the average velocity of this particle as t goes from 0 to $\pi/2$?

A) $\frac{5}{\pi}$ B) $\frac{10}{\pi}$ C) 5π D) 10π E) NOTA

23. When using an equation to describe a given shape graphically, you can often represent it easily in either rectangular or polar coordinates by making use of the identities $y=r\sin\theta$, $x=r\cos\theta$, and of course $x^2+y^2=r^2$. Knowing this, see if you can find the area of the region bounded by the polar graphs of $r=\frac{\sin\theta}{\cos^2\theta}$, $r=\sec\theta$, and $\theta=0$.

A) $\frac{2\pi}{3}$ B) $\frac{\pi}{3}$ C) $\frac{2}{3}$ D) $\frac{1}{3}$ E) NOTA

24. Evaluate in terms of a: $\int_{e^{-a}}^{e^{a}} \frac{dx}{x(x+1)}$

A) a B) e^a C) $3a$ D) $3+2\ln\left(e^a+1\right)$ E) NOTA

25. Let $P(x)$ be the unique 3rd-degree polynomial that, for any positive integer x, satisfies: $P(x) = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + ... + x^2$. Evaluate: $\frac{d^3(P(x))}{dx^3}$

A) 2 B) 0 C) 6 D) x^2 E) NOTA

26. $\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{\pi-e}{n}\right) \ln\left(e+\frac{k(\pi-e)}{n}\right)^{\left(e+\frac{k(\pi-e)}{n}\right)} =$

A) $\pi^2\left(\ln\pi^2-1\right)-e^2$ B) ∞ C) $\frac{\pi^2\left(\ln\pi^2-1\right)-e^2}{4}$ D) $\frac{\pi^2\ln\pi-e^2}{2}$ E) NOTA

27. Sue Madre stands at the point (1, 1) on the graph of $y=x^2$. She walks a short distance in the +x-direction, and then turns and walks in the +y-direction until she meets again with the graph of $y=x^2$. As the distance she walks becomes less and less, what is the ratio of how far she walks horizontally to how far she would've walked had she just walked along $y=x^2$?

A) $\frac{\sqrt{5}}{5}$ B) 1 C) 2 D) $\sqrt{5}$ E) NOTA

28. How many distinct real numbers satisfy the Mean Value Theorem for Derivatives for $f(x)=\sin\left(\frac{1}{x}\right)$ on $\left(\frac{1}{7\pi},\frac{1}{2\pi}\right)$?

A) 3 B) 4 C) 5 D) 6 E) NOTA

29. If $\frac{dy}{dx}=xy$ and $y(0)=e$, what is $y(2)$?

A) 1 B) e^2 C) $2e^3$ D) e^{2+e} E) NOTA

30. The expression $\lim_{h\to 0}\frac{f(x+h)-f(h)}{x-h}$ represents the slope of:

A) The line between $\left(x,f(x)\right)$ and the origin. B) The line between $\left(x,f(x)\right)$ and $\left(0,f(0)\right)$.
C) The tangent line to f at x. D) The normal to f at x. E) NOTA

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Calculus Team Round
Condensed Version

1. What is the first English word (alphabetically) that can be made using all of the letters below that are next to expressions that are equal to π?

A: $4-\frac{4}{3}+\frac{4}{5}-\frac{4}{7}+\frac{4}{9}-\frac{4}{11}+\ldots$

B: $3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\cfrac{1}{292+\cfrac{1}{1+\frac{1}{\sqrt{3}}}}}}}$

C: $\int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}}$

D: $20\tan^{-1}\left(\frac{1}{7}\right)+8\tan^{-1}\left(\frac{3}{79}\right)$

E: $2\ln i^{-i}$

F: $\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{3}+\frac{\sqrt{3}}{5}+\frac{\sqrt{3}}{7}+\frac{\sqrt{3}}{11}+\ldots$

2. Let $f(n)=\frac{c^n-(1-c)^n}{2c-1}$ for some real constant c, with the properties that for all real n, $f(n)=f(n-1)+f(n-2)$, and $f(3)=2$. What is $\int_{f(20)-f(18)}^{f(19)} c^x\,dx$?

3. Let $A=\int_0^1 x^x(\ln x+1)dx$. Let $B=\int_e^{e^2}\frac{dx}{x\ln x}$. What is $\int_A^B \begin{vmatrix} 2\sin x & x \\ \ln x & 2\cos x\end{vmatrix} dx$ to the nearest thousandth?

4. You shouldn't have been able to get through a year in Calculus if you can't integrate the Riemann way:

Let A = $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\pi}{n}\sin\left(\frac{k\pi}{n}\right)$

Let B = $\lim_{n\to\infty}\sum_{k=1}^{n}15\sqrt{n+15k}\cdot n^{-\frac{3}{2}}$

Let C = $\lim_{n\to\infty}\sum_{k=1}^{n}\ln\left(1+\frac{k(e-1)}{n}\right)^{\left(\frac{e-1}{n}\right)}$

What is $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{B-A}{n}\left(A+\frac{k(B-A)}{n}\right)^{C}$?

5. What is the slope of the line that is tangent to the graph of $y=x^4-6x^3-16x^2+54x+63$ in two places?

6. Let $A=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$, $B=\lim_{x\to 0}\left(1+\frac{1}{x}\right)^x$, $C=\lim_{x\to\infty}(1+x)^{\frac{1}{x}}$, $D=\lim_{x\to 0}(1+x)^{\frac{1}{x}}$. What is $ABCD$?

7. Let A = the volume formed by revolving the area between x^2 and $2x$ around the y-axis.
Let B = the volume formed by revolving the area between x^2 and $2x$ around the x-axis.
What is $A + B$?

8. What is the volume (to the nearest thousandth) formed by revolving the graph of $x^2 + y^2 = 1$ where $y \geq -\frac{\sqrt{3}}{2}$ around the line $y = -\frac{\sqrt{3}}{2}$?

9. Let A = the region made up of all points (x, y) satisfying $y > x$, $y > x^2$, and $y \leq 1$.
Let N = the region made up of all points (x, y) satisfying $y < x$, $y > -\sqrt{x}$, and $x \leq 1$.
Let D = the region made up of all points (x, y) satisfying $y < x^3$, $y < -\sqrt{x}$, and $y \geq -1$.
Let Y = the region made up of all points (x, y) satisfying $y < x^2$, $y > x^3$, and $x \geq -1$.
What is the perimeter of the region $A \cup N \cup D \cup Y$?

10. Let $f(a,b) = \int_a^b \left|\sin\frac{1}{x}\right| dx$. What is $f\left(\frac{1}{\pi}, \frac{1}{2\pi}\right) + f\left(\frac{1}{3\pi}, \frac{1}{4\pi}\right) + f\left(\frac{1}{2\pi}, \frac{1}{3\pi}\right) + f\left(\frac{1}{4\pi}, \frac{1}{5\pi}\right)$ to the nearest thousandth?

11. For some constants $n \neq 0$, $a > 0$, let $f(x) = \frac{x^{n+1}}{2(n+1)}$, $g(x) = \frac{1}{2(n-1)x^{n-1}}$, and $h(x) = f(x) + g(x)$. In terms of only f, g, a, and universal constants, what is the arc length of $h(x)$ from 0 to a?

12. If $a(x) = x^x$, $b(x) = x^{x^x}$, and $y(x) = x^{x^{x^{\cdots}}}$, what is $a'(e) + b'(1) + y'\left(\sqrt{2}\right)$ to the nearest thousandth?

13. What is the second smallest positive integer that leaves a remainder of 2 when divided by 3, 5 when divided by 7, 11 when divided by 13, and 17 when divided by 19?

14. Let $t = \frac{10}{i \cdot \sum_{n=1}^{\infty} \frac{F(n)}{10^n}}$, where $F(n)$ is the nth Fibonacci number ($F(1) = 1$, $F(2) = 1$, $F(3) = 2$, etc.).
What is $ti - 89$?

15. The number of ways of choosing r people from a group of n is $\sum_{x=0}^{100} \binom{100}{x}^2$, where $\binom{a}{b} = \frac{a!}{b!(a-b)!}$. What is $n + r$?

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Answer Key

1. A
2. D
3. B
4. C
5. C
6. D
7. E $-\frac{1}{144}$
8. A
9. E $\frac{4r}{3}$
10. E $\pi-\sqrt[4]{\frac{2143}{22}}$
11. B
12. A
13. C
14. C
15. A
16. B
17. B
18. B
19. B
20. E
21. C
22. B
23. D
24. A
25. A
26. C
27. A
28. C
29. E e^3
30. B

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Answer Key

1. ACED
2. 0
3. 1.025
4. 880
5. –21
6. e^2
7. $\frac{104\pi}{15}$
8. 17.125
9. 8
10. .156
11. f(a)-g(a)
12. 40.526
13. 0
14. 2.607
15. 300

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Solutions

1. Nested functions' even/oddness works like multiplication with integers. E*E = E*O = O*E = E, O*O = O. Thus the given composition is always even. **A!**

2. Since triangle ABC's base and height are remaining constant, so is the area. 0! **D!**

3. The graph is a square with vertices at (0, 1), (1, 0), (0, -1), and (-1, 0). Revolving forms two right circular cones, each with height and radius 1 (and volume $\frac{\pi}{3}$), so the total volume is $\frac{2\pi}{3}$! **B!**

4. The degree version is just the radian version stretched horizontally by a factor of $\frac{180}{\pi}$! **C!**

5. The solid formed is a sphere with radius 1 and volume $\frac{4\pi}{3}$! **C!**

6. The given Riemann sum is actually the integral $\int_a^b x \cdot dx = \left[\frac{x^2}{2}\right]_a^b = \frac{b^2 - a^2}{2}$! **D!**

7. Originally, they sold x cans for x dollars. Now they will be selling $x + 1$ cans for x dollars, and must charge $f(x) = \frac{x+1}{x} = 1 + \frac{1}{x}$ dollars per can to compensate. $f'(12) = -\frac{1}{144}$! **E!**

8. Out of the first 10^n pages, only the first 10% have less digits than the last 90%, which acquire more and more digits to infinity as n goes to infinity. Thus the average digits of the first 10^n pages will be at least 90% of the number of digits on the last 90% of the pages, and everything goes to infinity! **A!**

9. $\int_0^2 \int_1^3 xy \cdot dx \cdot dy = \int_0^2 \frac{x^2}{2}\Big|_1^3 y \cdot dy = \int_0^2 4y \cdot dy = 2y^2\Big|_0^2 = 8$. **E!**

10. It turns out there's nothing undefined or sketchy at all about this limit. Simply plugging in $x = \sqrt{2}$ yields the answer $\pi - \sqrt[4]{\frac{2143}{22}}$! This turns out to be extremely small because it utilizes Ramanujan's approximation $\pi \approx \sqrt[4]{\frac{2143}{22}} = 3.1415926526...$ **E!**

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Solutions

11. Taking a derivative of the equation just the way you see it yields $-\frac{f'\left(\frac{1}{x}\right)}{x^2} = -\frac{1}{(x+1)^2}$. Plugging in $x = 2$ gives the answer $f'\left(\frac{1}{2}\right) = \frac{4}{9}$! OR Substitute $\frac{1}{x} \to x$ and find $f(x) = \frac{1}{\frac{1}{x}+1} = \frac{x}{x+1} = 1 - \frac{1}{x+1}$. Then $f'(x) = \frac{1}{(x+1)^2}$ so $f'\left(\frac{1}{2}\right) = \frac{4}{9}$! **B!**

12. The trick is that they're both moving with speed 1 along their respective graphs. Since Bofa is moving in a straight line, there's no way Dienda can catch him without going faster than him. 0! **A!**

13. Substitute $u = \tan x$, $du = \sec^2 x \cdot dx$: $\int_0^{\tan^{-1}\sqrt{2}} \frac{du}{\cos u} = \left[\ln\left|\frac{\cos x}{\sin x - 1}\right|\right]_0^{\tan^{-1}\sqrt{2}} = \ln\left(\sqrt{2}+\sqrt{3}\right)$! **C!**

14. The region is an equilateral triangle with vertices at the origin and $\left(\pm\sqrt{3},\ 3\right)$. The centroid of a triangle is the average of its coordinates: $\left(\frac{0+0+0}{3},\ \frac{0+3+3}{3}\right) = (0,\ 2)$! **C!**

15. Anytime you have more than 365 people in a room, you are guaranteed to have at least 2 with the same birthday (in this case at least 6; see Pigeonhole Principle)! Thus $P(2005) - P(2004) = 1 - 1 = 0$! **A!**

16. The area of the semi-circle is $\frac{\pi}{2}$. The area of the "cone" (isosceles triangle) is 3. **B!**

17. The semi-circle forms a hemisphere of radius 1 and volume $\frac{2\pi}{3}$. The triangle forms a cone of radius 1, height 3, and volume π. **B!**

18. Take the $\frac{\pi+6}{2}$ from 16 and add in the new circle of area π, giving $\frac{3\pi+6}{2}$. Now subtract the overlap between the two "scoops," which has area $2\left(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right) = \frac{2\pi}{3}-\frac{\sqrt{3}}{2}$.

$\frac{3\pi+6}{2} - \left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}+3+\frac{\sqrt{3}}{2}$! **B!**

19. Take a $g(\ )$ of both sides of the given equation, then take a derivative of the result with respect to x: $g\left(f(x)^2\right) = 4x - 1$, $2f'(x)f(x)g'\left(f(x)^2\right) = 4$. Dividing both sides by 2 shows the desired quantity equals 2! **B!**

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20. Graph the inequality in the *ab*-plane, and we see the probability is $\dfrac{\int_0^2 a^2 da}{\int_0^2 4\,da} = \dfrac{\frac{8}{3}}{8} = \dfrac{1}{3} \approx .33$! **E!**

21. $\int_0^t \left(x^n - x^{n+1}\right)dx = \dfrac{t^{n+1}}{n+1} - \dfrac{t^{n+2}}{n+2} = 0 \rightarrow t = 0,\ 1 + \dfrac{1}{n+1}$! **C!**

22. $\dfrac{\sqrt{\left(x\left(\frac{\pi}{2}\right) - x(0)\right)^2 + \left(y\left(\frac{\pi}{2}\right) - y(0)\right)^2}}{\frac{\pi}{2} - 0} = \dfrac{10}{\pi}$! **B!**

23. $r = \dfrac{\sin\theta}{\cos^2\theta} \rightarrow r^2\cos^2\theta = r\sin\theta = y = x^2$. $r = \sec\theta \rightarrow r\cos\theta = x = 1$. $\theta = 0 \rightarrow y = 0$. The area of the region bound by $y = x^2$, $x = 1$, and $y = 0$ is $\int_0^1 x^2 dx = \dfrac{1}{3}$! **D!**

24. $\int_{e^{-a}}^{e^a} \dfrac{dx}{x(x+1)} = \int_{e^{-a}}^{e^a} dx\left(\dfrac{1}{x} - \dfrac{1}{x+1}\right) = \left[\ln x - \ln(x+1)\right]_{e^{-a}}^{e^a} = \ln e^a - \ln\left(e^a + 1\right) - \left(\ln e^{-a} - \ln\left(e^{-a}+1\right)\right) =$ **A!**
$= a - \ln\left(e^a + 1\right) + a + \ln\left(\dfrac{e^a + 1}{e^a}\right) = 2a - \ln\left(e^a+1\right) + \ln\left(e^a + 1\right) - a = a.$

25. $P(x) = \dfrac{x(x+1)(2x+1)}{6} = \dfrac{x^3}{3} + \dfrac{x^2}{2} + \dfrac{x}{6}$. $\dfrac{d^3(P(x))}{dx^3} = 2$! **A!**

26. Using the substitutions $dx = \left(\dfrac{\pi - e}{n}\right)$ and $x = \left(e + \dfrac{k(\pi - e)}{n}\right)$ we arrive at the integral $\int_e^\pi x\ln x \cdot dx = \left[\dfrac{x^2}{2}\ln x - \dfrac{x^2}{4}\right]_e^\pi = \dfrac{\pi^2\left(\ln\pi^2 - 1\right) - e^2}{4}$! **C!**

27. At the point (1, 1), the graph of $y = x^2$ has a slope of 2. So as she runs less, her path approaches a $1 - 2 - \sqrt{5}$ right triangle, the 1 being horizontal, the 2 vertical, and the $\sqrt{5}$ diagonal along the graph. $\dfrac{1}{\sqrt{5}} = \dfrac{\sqrt{5}}{5}$! **A!**

28. It's the same as doing $\sin x$ on $(2\pi,\ 7\pi)$. There is one solution per "hump" (π), for a total of 5! **C!**

29. $\dfrac{dy}{dx} = xy \rightarrow \dfrac{dy}{y} = x \cdot dx \rightarrow \ln|y| = \dfrac{x^2}{2} + C$. Plugging in yields $C = 1$. $y(2) = e^3$! **E!**

30. Just plug in $h = 0$, nothing is undefined: $\dfrac{f(x) - f(0)}{x - 0}$. **B!**

Calculus

March Regional – Team Round Solutions

1. A is 4 times the Taylor Series for $\tan^{-1}x$ at $x = 1$, which is equal to $\frac{\pi}{4}$. B is a good approximation, but since pi is transcendental, no finite combination of rational and irrational numbers can equal it. Integrating, C is $\cos^{-1}(-1) - \cos^{-1}(1) = \pi$. D and E work (use your calculator) and F doesn't. The word is ACED.

2. By definition, $f(20) = f(19) + f(18)$, so the two bounds on the integrand are equal. 0!

3. $\int_0^1 x^x(\ln x + 1) = x^x\Big|_0^1 = 0$. $\int_e^{e^2} \frac{dx}{x\ln x} = \ln(\ln x)\Big|_e^{e^2} = \ln 2$.
$\int_0^{\ln 2} (4\sin x\cos x - x\ln x) \bullet dx = \frac{x^2}{4} - \frac{x^2}{2}\ln x - (\cos 2x)\, from\, 0\, to\, \ln 2 \approx 1.025$

4. $A = \int_0^{\pi} \sin x dx = 2$. $B = \int_1^{16} \sqrt{x} dx = 42$. $C = \int_1^{e} \ln x dx = 1$. $Answer = \int_A^B x^C\, dx = \frac{B^{C+1} - A^{C+1}}{C+1} = 880$.

5. The polynomial $x^4 - 6x^3 - 16x^2 + 54x + 63 - mx - b$, where m and b represent the slope and y-intercept of the line, must have 2 double roots, i.e. be the square of a quadratic equation, of the form: $(x^2 + cx + d)^2 = x^4 + 2cx^3 + (2d + c^2)x^2 + 2cdx + d^2$. Equating the coefficients: $-6 = 2c$, $-16 = 2d + c^2$, $54 - m = 2cd$, and $63 - b = d^2$ yields $m = -21$.

6. You can use a TI-89: $A = D = e, B = C = 1, ABCD = e^2$!

7. $A = 2\pi\int_0^2 x(2x - x^2)dx = \frac{8\pi}{3}$. $B = \pi\int_0^2 \left((2x)^2 - (x^2)^2\right) = \frac{64\pi}{15}$. $A + B = \frac{104\pi}{15}$.

8. $\pi\int_{-1}^{1}\left(\sqrt{1-x^2} + \frac{\sqrt{3}}{2}\right)^2 dx - 2\cdot\pi\int_{\frac{1}{2}}^{1}\left(-\sqrt{1-x^2} + \frac{\sqrt{3}}{2}\right)^2 dx = \frac{\pi(10\pi\sqrt{3} + 11)}{12}$. The first integral rotates the entire top half of the circle, and the second integral rotates the two equal sections between the bottom half and the line, from (-1, -1/2) to (1/2, 1) – I just did the latter and doubled it above. An alternate method is to slice do quadrants I and IV, split at $x = 1/2$, then double it:

$$2\pi\int_{\frac{1}{2}}^{1}\left(\left(\sqrt{1-x^2} + \frac{\sqrt{3}}{2}\right)^2 - \left(-\sqrt{1-x^2} + \frac{\sqrt{3}}{2}\right)^2\right)dx + 2\pi\int_0^{\frac{1}{2}}\left(\sqrt{1-x^2} + \frac{\sqrt{3}}{2}\right)^2 dx = \frac{\pi(10\pi\sqrt{3} + 11)}{12} \approx 17.125.$$

9. The union is just the square with vertices at $(\pm 1, \pm 1)$, cut into four strange regions. So the perimeter is 8.

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10. Answer = $\int_{\frac{1}{5\pi}}^{\frac{1}{\pi}} \left|\sin\frac{1}{x}\right| dx \approx .156.$

11.

$$\int_0^a \sqrt{1+\left(\frac{d\left(\frac{x^{n+1}}{2(n+1)}+\frac{1}{2(n-1)x^{n-1}}\right)}{dx}\right)^2}\,dx = \int_0^a \sqrt{1+\left(\frac{x^n}{2}-\frac{1}{2x^n}\right)^2}\,dx = \int_0^a \sqrt{\frac{x^{2n}}{4}+\frac{1}{2}+\frac{1}{4x^{2n}}}\,dx = \int_0^a \left(\frac{x^n}{2}+\frac{1}{2x^n}\right)dx =$$

$$= \frac{x^{n+1}}{2(n+1)} - \frac{1}{2(n-1)x^{n-1}}\Bigg|_0^a = f(a) - g(a).$$

12. $a'(x) = x^x(\ln x + 1)$. $b'(x) = x^{x^x+x-1}(1 + x\ln x(\ln x + 1))$.

$y = x^y \Rightarrow \ln y = y\ln x \Rightarrow \frac{y'}{y} = y'\ln x + \frac{y}{x} \Rightarrow y' = \frac{y^2}{x - xy\ln x}$. $y(\sqrt{2}) = 2, so\ y'(\sqrt{2}) = \frac{2\sqrt{2}}{1-\ln 2} = 9.21754.$

$a'(e) + b'(1) + 9.21754 \approx 40.526.$

13. See Chinese Remainder Theorem. Or, notice that the last three conditions can be satisfied by the number that is 2 less than the LCM of 7, 13, and 19, i.e. 1729 – 2 = 1727. Since 1727 happens to also satisfy the first condition, it is the smallest positive number to satisfy all 4, and the process has a period of LCM(3,7,13,19) = 5187. So the second smallest number is 1727 + 5187 = 6914.

14. Using the formula for the *n*th Fibonacci number and the formula for the sum of a geometric series:

$$\sum_{n=1}^{\infty}\frac{F(n)}{10^n} = \sum_{n=1}^{\infty}\frac{\frac{\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}}{10^n} = \frac{1}{\sqrt{5}}\left(\sum_{n=1}^{\infty}\left(\frac{1+\sqrt{5}}{20}\right)^n - \sum_{n=1}^{\infty}\left(\frac{1-\sqrt{5}}{20}\right)^n\right) = \frac{1}{\sqrt{5}}\left(\frac{\frac{1+\sqrt{5}}{20}}{1-\frac{1+\sqrt{5}}{20}} - \frac{\frac{1-\sqrt{5}}{20}}{1-\frac{1-\sqrt{5}}{20}}\right) = \frac{10}{89}$$

$t = \frac{10}{i\cdot\frac{10}{89}} = -89i$. $ti - 89 = 89 - 89 = 0$! (You can plug this directly into a *ti*-89!)

15. In general, $\sum_{x=0}^{n}\binom{n}{x}^2 = \binom{2n}{n}$. Since $\binom{n}{x} = \binom{n}{n-x}$, think of $\binom{n}{x}^2 = \binom{n}{x}\binom{n}{n-x}$. This is the number of ways to choose *x* people from a group of *n*, times the number of ways to choose *n* – *x* people from a group of *n*. Over all valid *x*, what you're really doing is choosing *x* + (*n* – *x*) = *n* people from a group of *n* + *n* = 2*n*. So the answer is since $\binom{200}{100}$, *n* = 200, *r* = 100, *n* + *r* = 300!