January 2006 Palm Harbor Invitational Calculus Individual Test

Let choice E) NOTA denote "None Of These Answers."

1) Determine: $\lim_{x \to -7} \frac{x^3 + 343}{x^2 - 49}$

A) -10.5 B) 0 C) 5.25 D) 10.5 E) NOTA

2) Find: $\int \cos^3(x)\sin(x)dx$

A) $\frac{1}{4}\cos^4(x) + C$ B) $\cos^4(x) + C$

C) $-\frac{1}{4}\cos^4(x) + C$ D) $-\cos^4(x) + C$ E) NOTA

3) Consider the area bounded by $y = |3x|$, $y = |x|$, $x = 6$, and $x = -6$. What is the volume of the solid formed by revolving this area about the x-axis?

A) 288π B) 576π C) 1152π D) 2304π E) NOTA

4) Given that $f(x) = (1+2x)^{6/x}$. By the $\varepsilon - \delta$ definition of a limit, $\lim_{x \to 0} f(x)$ means that $\forall \varepsilon > 0\ \exists \delta > 0$ such that $\forall x \in \mathbb{R}\ \ 0 < |x - 0| < \delta$ implies that:

A) $|f(x) - \varepsilon| < e^{1/3}$ B) $\left|f(x) - e^{1/3}\right| < \varepsilon$

C) $|f(x) - \varepsilon| < e^3$ D) $\left|f(x) - e^3\right| < \varepsilon$ E) NOTA

5) Given $y = 3y^2 - 5xy + 2x^2 - 13$. What is $\frac{d^2y}{dx^2}$ where $x = 3$ and $y > 1$?

A) $\frac{11}{1372}$ B) $\frac{31}{42}$ C) $\frac{13}{14}$ D) $\frac{3074}{343}$ E) NOTA

6) Determine (to the ten-thousandths): $\lim_{n \to \infty} \sum_{i=1}^{n} \frac{\ln\left(\frac{n+i}{n}\right)}{n+i}$

A) 0.2402 B) 0.4804 C) 0.4805 D) 0.6931 E) NOTA

7) A particle moves along the x-axis with a constant acceleration $a(t) = 4$ and initial conditions of $v(0) = -2$ and $s(0) = -1$ for the velocity and position functions, respectively. Which of the following will give the total distance traveled by the particle over the interval $[0,10]$?

A) $s(10) - s(0)$ B) $\int_0^{10} s(t)dt$ C) $\int_0^{10} v(t)dt$ D) $\int_0^{10} |v(t)|\,dt$ E) NOTA

8) Marshall has enough money to buy only 300 square yards of sheet metal with which to create a cylindrical tank (such that the tank is open at only one end). What is the volume of the largest such tank Marshall can create with his limited resources (in cubic feet to the nearest hundredth)?

A) 20567.58 B) 15233.12 C) 7334.46 D) 16.93 E) NOTA

9) How many asymptotes does $f(x)$ have?

$$f(x) = \frac{x+2}{\sqrt{x^2 + x - 6}}$$

A) 1 B) 2 C) 3 D) 4 E) NOTA

10) Given: $f(0) = -3$ $f'(x) = 6x + 5$

Estimate one of the roots (to the nearest thousandths) of $f(x)$ using Newton's Method with two iterations, starting with $x_0 = 0$.

A) 0.468 B) 0.474 C) 0.480 D) 0.486 E) NOTA

11) What is the equation for a tangent line to the first derivative of $g(x) = \frac{5^x}{\ln x}$ at $x = 5$ with coefficients and constants rounded to the hundredths?

A) $y = 2883.71x - 12476.88$ B) $y = 4361.05x - 18921.54$
C) $y = 2883.72x - 12476.93$ D) $y = 4361.05x - 18921.53$ E) NOTA

12) The intersection of $x^2 + y^2 = 784$ and $y = |x|$ creates two areas, A_1 and A_2, where $A_1 < A_2$. Find $\frac{A_1}{A_2}$.

A) $\frac{1}{4}$ B) $\frac{1}{3}$ C) $\frac{2}{3}$ D) $\frac{3}{4}$ E) NOTA

13) How many points of inflection for the graph of $y = \sqrt{\cos(7x)}$ are there on the interval $(3.815, 5.161)$?

A) 1 B) 2 C) 3 D) 4 E) NOTA

For Questions 14-16 consider a vector-valued function $r(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$. If $r(t)$ is a position vector for a particle at parameter t, $r'(t)$ may be interpreted as a velocity vector for that particle at parameter t and $r'(t) = f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k}$.

For Questions 14-16 consider the vector-valued function $s(t) = \left(3t^2 + \frac{4}{t}\right)\vec{i} + \vec{j} + (\ln t)^2\,\vec{k}$ to describe the position of a particle in space.

14) What is the velocity vector of this particle at $t = 2$?

A) $11\vec{i} + \ln 2\vec{k}$ B) $13\vec{i} + \ln 2\vec{k}$ C) $13\vec{i} + 2\ln 2\vec{k}$ D) $11\vec{i} + 2\ln 2\vec{k}$ E) NOTA

15) What is the speed of this particle at $t = 3$ (to the nearest thousandths)?

A) 17.571 B) 17.598 C) 17.599 D) 17.600 E) NOTA

16) Extend the above explanation of vector-valued functions to find an acceleration vector for this particle at $t = 1$.

A) $-2\vec{i}$ B) $14\vec{i}$ C) $-2\vec{i} + 2\vec{k}$ D) $14\vec{i} + 2\vec{k}$ E) NOTA

17) Find $f'(x)$, if $f(x) = \left(x \ln|\sin x|\right)^3$.

A) $f'(x) = 3\left(x \ln|\sin x|\right)^2$

B) $f'(x) = 3\left(x \ln|\sin x|\right)^2 \left(\ln|\sin x| + \dfrac{x}{\sin x}\right)$

C) $f'(x) = \left(x \ln|\sin x|\right)^2 \left(\ln|\sin x| + x\cos x\right)$

D) $f'(x) = 3\left(x \ln|\sin x|\right)^2 \left(\ln|\sin x| + x\cot x\right)$

E) NOTA

18) Determine: $$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^4}{n^5}$$

A) 0 B) $\frac{1}{5}$ C) $\frac{1}{4}$ D) 1 E) NOTA

19) The rate at which a duck quacks is inversely proportional to the cube of the number of the number of quacks. If at time $t=2$ the duck quacks 2 times and at $t=0$ the duck quacks 0 times, what is the rate at which the duck quacks at $t=32$?

A) 0.03125 B) 0.0625 C) 2 D) 4 E) NOTA

20) Determine: $$\int_2^4\int_{2y}^{y^2} 6\,dx\,dy$$

A) 24 B) 32 C) 40 D) 48 E) NOTA

21) Find the area of the following polar graph: $r=12\cos\left(\theta+12^\circ\right)$

A) 9π B) 36π C) 144π D) 576π E) NOTA

22) Jimmy is pretending that the Epcot geosphere is a perfect sphere and measures it's diameter to be 65.93 meters. His measurement is correct to within 1 centimeter. Which of the following intervals represents the possible values of the volume of the Epcot geosphere if Jimmy is considering his propagated error in calculating the volume (interval endpoints to the nearest hundredths in meters cubed)?

A) $[136398.33, 163709.86]$ B) $[143226.21, 156881.98]$
C) $[149917.54, 150190.65]$ D) $[149985.82, 150122.37]$ E) NOTA

23) Determine: $$\lim_{x\to 0}\left(x^{-1}\tan(2x)\sin^4(x)-x^{-1}\tan(2x)\cos^4(x)\right)$$

A) 2 B) 0 C) -1 D) $-\frac{1}{2}$ E) NOTA

24) Find d, where $\int_0^{15}|x-d|\,dx=9$ and $0<\frac{1}{15}<\frac{1}{d}$.

A) 1 B) 2 C) 3 D) 4 E) NOTA

25) Consider the region bounded by the ellipse $x^2 + 4y^2 = 144$. A solid is formed by taking cross sections perpendicular to the y-axis that are rhombi, where the longer diagonal of the rhombi has its endpoints on the ellipse. One interior angle of all the rhombi cross sections is 60 degrees. What is the volume of this solid?

A) $48\sqrt{3}$ B) $384\sqrt{3}$ C) $768\sqrt{3}$ D) $2304\sqrt{3}$ E) NOTA

26) Determine: $\displaystyle\lim_{x\to 1}\frac{\int_1^x \ln\left(ex^2\right)}{\sqrt{x}-1}$

A) 1 B) 2 C) 4 D) Does not exist E) NOTA

27) What is the length of the smallest interval around $x = \pi$ that guarantees the existence of the horizontal tangent line to $y = \cos x$ at $x = \pi$ by the Mean Value Theorem?

A) 1 B) $\frac{\pi}{2}$ C) π D) 2π E) NOTA

28) $y = \cfrac{-8}{x + \cfrac{8}{x + \cfrac{8}{x + \dots}}}$, $x, y > 0$ Find $\frac{dy}{dx}$ where $x = 2$.

A) 4 B) 2 C) $\frac{3}{2}$ D) $\frac{2}{3}$ E) NOTA

29) What is the area of the region bounded by the graphs of $f(x) = x^2 - 5x$ and $g(x) = 4x^3 - 15x^2 - 25x$?

A) 288 B) $\frac{875}{3}$ C) $\frac{886}{3}$ D) 296 E) NOTA

30) What is the value of the 7th derivative of $f(x)$ at $x = 2$?

$$f(x) = 2x^8 + 11x^7 + 31x^6 + 29x^5 + 13x^4 + 4x^3 + 7x^2 + 17x + 1$$

A) 80640 B) 136080 C) 158400 D) 216720 E) NOTA

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1) **A** $\lim_{x\to -7}\frac{x^3+343}{x^2-49}=\lim_{x\to -7}\frac{(x+7)(x^2-7x+49)}{(x-7)(x+7)}$

$=\lim_{x\to -7}\frac{(x^2-7x+49)}{(x-7)}=-10.5$

2) **C** $\int \cos^3(x)\sin(x)dx=-\frac{1}{4}\cos^4(x)+C$

3) **C** Solid is formed by two cones. Difference in cone volumes in volume of solid.
Big Cone: $r=18, h=6$
Small Cone: $r=6, h=6$

$V_{solid}=2\left(V_{BigCone}-V_{SmallCone}\right)$

$=2\left(\frac{1}{3}\pi 18^2(6)-\frac{1}{3}\pi 6^2(6)\right)=1152\pi$

4) **D** $y=\lim_{x\to 0}(1+2x)^{6/x}$

$\ln y=\lim_{x\to 0}\ln(1+2x)^{6/x}$

$\ln y=\lim_{x\to 0}\frac{6\ln(1+2x)}{x}$ Use L'Hopital's.

$\ln y=\lim_{x\to 0}\frac{6}{2}=3$

$y=e^3\Rightarrow \lim_{x\to 0}(1+2x)^{6/x}=e^3$

Answer choice D is in correct form.

5) **A** $y=3y^2-5xy+2x^2-13$

$\frac{dy}{dx}=\frac{4x-5y}{5x+1-6y}$

$\frac{d^2y}{dx^2}=\frac{\left(4-5\frac{dy}{dx}\right)(5x+1-6y)-(4x-5y)\left(5-6\frac{dy}{dx}\right)}{(5x+1-6y)^2}$

$x=3\Rightarrow y=5, 1/3\Rightarrow y=5$ for $y>1$

$x=3, y=5\Rightarrow \frac{dy}{dx}=\frac{13}{14}$

$x=3, y=5, \frac{dy}{dx}=\frac{13}{14}\Rightarrow \frac{d^2y}{dx^2}=\frac{11}{1372}$

6) **A** $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\ln\left(\frac{n+i}{n}\right)}{n+i}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{\ln\left(1+\frac{i}{n}\right)}{1+\frac{i}{n}}$

Limit is a Riemann Sum for:

$\int_1^2\frac{\ln x}{x}dx=\left[\frac{1}{2}(\ln x)^2\right]_1^2=.2402$

7) **D** A and C give the net displacement of the particle along the *x*-axis.
B has no physical interpretation.

8) **B** 300 square yards = 2700 square feet

$A_{OpenCylinder}=2700=\pi r^2+2\pi rh$

$h=\frac{2700-\pi r^2}{2\pi r}$

$V=\pi r^2h=\pi r^2\frac{2700-\pi r^2}{2\pi r}=\frac{2700r-\pi r^3}{2}$

$V'=\frac{2700-3\pi r^2}{2}=0\Rightarrow r=16.9257$

$V=\frac{2700(16.93)-\pi(16.93)^3}{2}=15233.12$

9) **D** $f(x)=\frac{x+2}{\sqrt{x^2+x-6}}=\frac{x+2}{\sqrt{(x+3)(x-2)}}$

Two vertical asymptotes: $x=2, x=-3$

$\lim_{x\to\infty}f(x)=1\quad \lim_{x\to-\infty}f(x)=-1$

Two horizontal asymptotes: $y=1, y=-1$

10) **B** Because $f'(x)=6x+5$ $f(0)=-3$
then $f(x)=3x^2+5x-3$.
Thus, Newton's Method:

$x_{n+1}=x_n-\frac{3x_n^{\ 2}+5x_n-3}{6x_n+5}$

$x_0=0\Rightarrow x_1=.6\Rightarrow x_2=.474$

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11) **B** $g(x)=\dfrac{5^x}{\ln x} \Rightarrow g'(x)=\dfrac{\ln 5\left(5^x\right)\ln x-\dfrac{5^x}{x}}{(\ln x)^2}$

$$g''(x)=\frac{\left(5^x\right)\left(\left(x^2(\ln 5)^2(\ln x)-2x(\ln 5)+1\right)(\ln x)+2\right)}{x^2(\ln x)^3}$$

$x=5 \quad g'(x)=2883.71 \quad g''(x)=4361.05$

$y-2883.71=4361.05(x-5)$

$y=4361.05x-18921.54$

12) **B** This is the graph of a circle with a quarter cut off by $y=|x|$.
Thus the ratio of the areas is 1 quarter to 3 quarters. This is $\dfrac{1}{3}$.

13) **B** $\cos x$ - points of inflection: $x=\dfrac{\pi}{2}+n\pi$

$\sqrt{\cos(7x)}$ - inflections pts: $x=\dfrac{\pi}{14}+n\dfrac{\pi}{7}$

$n=9,10$ give values in interval

2 points of inflection

14) **A** $s(t)=\left(3t^2+\dfrac{4}{t}\right)\vec{i}+\vec{j}+(\ln t)^2\vec{k}$

$s'(t)=\left(6t-\dfrac{4}{t^2}\right)\vec{i}+\dfrac{2(\ln t)}{t}\vec{k}$

$s'(2)=11\vec{i}+\ln 2\vec{k}$

15) **A** $s'(t)=\left(6t-\dfrac{4}{t^2}\right)\vec{i}+\dfrac{2(\ln t)}{t}\vec{k}$

$s'(3)=17.556\vec{i}+0.732\vec{k}$

Speed equals length of velocity vector:

$\|s'(t)\|=\sqrt{17.556^2+0.732^2}=17.571$

16) **D** $s'(t)=\left(6t-\dfrac{4}{t^2}\right)\vec{i}+\dfrac{2(\ln t)}{t}\vec{k}$

$s''(t)=\left(6+\dfrac{8}{t^3}\right)\vec{i}+\left(-\dfrac{2(\ln t)}{t^2}+\dfrac{2}{t^2}\right)\vec{k}$

$s''(1)=14\vec{i}+2\vec{k}$

17) **D** $f(x)=\left(x\ln|\sin x|\right)^3$

$f'(x)=3\left(x\ln|\sin x|\right)^2\left(\ln|\sin x|+\dfrac{x}{\sin x}\cos x\right)$

$f'(x)=3x^2\left(\ln|\sin x|\right)^3+3x^3\left(\ln|\sin x|\right)^2\cot x$

18) **B** $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^4}{n^5}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^4$

Limit is a Riemann Sum for:

$\displaystyle\int_0^1 x^4\,dx=\left[\frac{1}{5}x^5\right]_0^1=\frac{1}{5}$

19) **A** $\dfrac{dQ}{dt}=\dfrac{k}{Q^3}\Rightarrow Q^3dQ=kdt$

$\displaystyle\int Q^3dQ=\int kdt\Rightarrow\frac{1}{4}Q^4=kt+C$

$C=0$ because $Q=0$ when $t=0$.

$k=2$ because $Q=2$ when $t=2$.

So $\dfrac{1}{4}Q^4=2t$ and $\dfrac{dQ}{dt}=\dfrac{2}{Q^3}$.

At $t=32$, then $Q=4$ and $\dfrac{dQ}{dt}=.03125$.

20) **C** $\displaystyle\int_2^4\int_{2y}^{y^2}6\,dx\,dy\Rightarrow\int_2^4\left(\int_{2y}^{y^2}6\,dx\right)dy\Rightarrow\int_2^4\left[6x\right]_{2y}^{y^2}dy$

$\displaystyle\Rightarrow\int_2^4\left(6y^2-12y\right)dy\Rightarrow\left[2y^3-6y^2\right]_2^4\Rightarrow 40$

21) **B** Graph is a circle of radius 6. $\pi r^2=36\pi$

22) **D** $diameter=65.93\Rightarrow radius=32.965$

1 centimeter = .01 meter

$error-diam=.01\Rightarrow error-rad\Rightarrow .005$

$V=\dfrac{4}{3}\pi r^3=\dfrac{4}{3}\pi(32.965)^3=150054.094$

$V=\dfrac{4}{3}\pi r^3\Rightarrow dV=4\pi r^2dr$

$=4\pi(32.965)^2(.005)=68.28$

$[150054.094-68.279, 150054.094+68.279]$

$=[149985.82, 150122.37]$

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23) **E** $x^{-1}\tan(2x)\sin^4(x)-x^{-1}\tan(2x)\cos^4(x)$

$$=\frac{\left(\sin^4(x)-\cos^4(x)\right)\tan(2x)}{x}$$

$$=\frac{\left(\sin^2(x)-\cos^2(x)\right)\left(\sin^2(x)+\cos^2(x)\right)\sin(2x)}{x\cos(2x)}$$

$$=-\frac{2\sin(2x)}{2x}\Rightarrow -2\lim_{x\to 0}\frac{\sin(2x)}{2x}=-2$$

24) **C** $\int_0^{15}|x-d|\,dx=9$

$$[-x+d]_0^d+[x-d]_d^{15}=9$$

$$15-2d=9$$

$$d=3$$

25) **C** $x^2+4y^2=144\Rightarrow x=\sqrt{144-4y^2}$

rhombus = two isosceles triangles

$$b_{triangle}=2\sqrt{144-4y^2}$$

$$h_{triangle}=\tan 30^\circ\sqrt{144-4y^2}=\sqrt{\frac{144-4y^2}{3}}$$

$$A_{CrossSection}=2\left(\frac{1}{2}bh\right)$$

$$=2\left(\frac{1}{2}\left(2\sqrt{144-4y^2}\right)\sqrt{\frac{144-4y^2}{3}}\right)$$

$$=\frac{8\left(36-y^2\right)}{\sqrt{3}}$$

$$V=2\int_0^6\frac{8\left(36-y^2\right)}{\sqrt{3}}dy$$

$$=\frac{16}{\sqrt{3}}\left[36y-\frac{1}{3}y^3\right]_0^6=768\sqrt{3}$$

26) **B** $\lim_{x\to 1}\frac{\int_1^x\ln\left(ex^2\right)}{\sqrt{x}-1}\times\frac{\sqrt{x}+1}{\sqrt{x}+1}$

$$=\lim_{x\to 1}\frac{\int_1^x\ln\left(ex^2\right)}{x-1}\times\lim_{x\to 1}\sqrt{x}+1$$

$$=\ln\left(e\times 1^2\right)\times 2=2$$

Limit is clever form of derivative.

27) **E** The length of the smallest interval is essentially 0, because a horizontal secant can be found going through the graph as arbitrarily close to the tangent line as desired.

28) **D** $y=\frac{-8}{x-y}$, $x=2$

$$y=\frac{-8}{2-y}\Rightarrow 0=y^2-2y-8\Rightarrow y=4$$

$$\frac{dy}{dx}=\frac{8\left(1-\frac{dy}{dx}\right)}{(x-y)^2}$$

$$x=2,\ y=4\Rightarrow\frac{dy}{dx}=\frac{2}{3}$$

29) **C** $f(x)=x^2-5x,\ g(x)=4x^3-15x^2-25x$

$$x^2-5x=4x^3-15x^2-25x$$

$$0=4x^3-16x^2-20x=4x(x-5)(x+1)$$

$$x=-1,0,5$$

$$\int_{-1}^0[g(x)-f(x)]dx+\int_0^5[f(x)-g(x)]dx=\frac{886}{3}$$

30) **D** $f^{(7)}(x)=80640x+55440$

$$f^{(7)}(2)=216720$$

2006 PHUHS January Invitational Calculus Team Question 1

A = the ratio of a circle's circumference to its radius.

$$B = \det\begin{bmatrix} 1 & 2 & 3 \\ 7 & -5 & 4 \\ -2 & 2 & -1 \end{bmatrix}$$

C = the smallest counting number that is the sum of its proper factors.
(note: proper factors of a number are all factors except the number itself.)

$$D = 2\int_{\lim\limits_{x\to 2^-}\left(\frac{x^2+2}{x-2}\right)}^{\ln(1)} \left(\frac{B-C}{\sqrt{A}} \left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x \right]^{\frac{-x^2}{2}} \right) dx$$

Find: D

2006 PHUHS January Invitational Calculus Team Question 2

A 6 foot tall man is 14 feet away from a 25 foot tall street light. He is walking away from the light at a rate of two feet per second.

A = rate at which the man's shadow is growing.

B = rate at which the tip of the man's shadow is receding from the light post.

Find: B - A

2006 PHUHS January Invitational Calculus Team Question 3

Let a = the volume in cubic inches of an open topped box formed by cutting squares out of the corners of a rectangular piece of metal whose dimensions are 18 inches by 24 inches.
A = the greatest integer less than or equal to the maximum value of a.
Let b = the surface area of a cylinder that would hold 12 ounces of soda.
B = the least integer greater than or equal to the minimum value of b.
(Note: 12 fluid ounces converts to 21.66 cubic inches.)

Find the lowest common multiple of A and B.

2006 PHUHS January Invitational Calculus Team Question 4

A conical tank (vertex down) has a height of 10 feet and a radius of 4 feet. Water is being poured into the tank at the rate of one cubic foot per minute.

A = how fast is the level of water rising in feet per minute when there is 50 cubic feet of water in the tank.

B = how fast the surface area of the circular top of water is expanding in square feet per minute when there is 50 cubic feet of water in the tank..

Find the product of A and B correct to the nearest thousandth.

2006 PHUHS January Invitational Calculus Team Question 5

A = $\int_{-4}^{2} \sqrt{16-x^2}\,dx$

B = $\int_{-2}^{-2+\sqrt{6}} \frac{x+2}{x^2+4x+6}\,dx$

Find: $A \times e^B$

2006 PHUHS January Invitational Calculus Team Question 6

A = area of one petal in r = $1-\cos(5\theta)$

B = area of the inside "petal" of r = $2-4\cos\theta$

C = area of an ellipse centered at the origin that passes through (5, 0) and the nearest focus to that point has coordinates (3, 0).

Find: $\frac{A \times B}{C}$ correct to the nearest thousandth

2006 PHUHS January Invitational Calculus Team Question 7

$$\text{Given: } f(x) = x^{x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}}$$

If $f(x)$ is a real valued function then the domain must be restricted to a certain range of positive real numbers. What is the maximum value for $f(x)$ on that interval?

2006 PHUHS January Invitational Calculus Team Question 8

Let P(x) be a continuous probability distribution function.

If $P(x) = \begin{cases} k\left(8x - x^3\right), 0 \le x \le 2 \\ 0, \qquad\qquad \text{otherwise} \end{cases}$, what is the value of k?

2006 PHUHS January Invitational Calculus Team Question 9

How many times greater is the probability of getting two pairs in a five card poker hand than getting a royal flush?

2006 PHUHS January Invitational Calculus Team Question 10

Triangle ABC has side AB = 8 inches and side AC = 10 inches. If angle A is 30 degrees at $t = 0$, and is increasing at 2 degrees per second, how fast is the area of the triangle increasing at $t = 0$?

2006 PHUHS January Invitational Calculus Team Question 11

A = $\lim\limits_{x\to-\infty}\left(\dfrac{x^2}{e^{-x}}\right)$, B = $\lim\limits_{x\to 0}\left(\dfrac{1-\cos 4x}{4x}\right)$, C = $\lim\limits_{x\to 2}\left(\cos\dfrac{\pi x}{3}\right)$, D = $\lim\limits_{x\to 3}\left(\dfrac{x^2+4x-21}{x-3}\right)$

Find: A + B + C + D

2006 PHUHS January Invitational Calculus Team Question 12

If a tie still exists, the order in which correct answers to this one are handed in will break the tie.

Find: $\dfrac{d}{dx}\left[\pi^3x^2+2\pi x-\pi\right]$

2006 PHUHS January Invitational Calculus Team Question 1

A = 2π B = 7 C = 6 ln(1) = 0 $\lim_{x\to 2^-}\left(\frac{x^2+2}{x-2}\right)=-\infty$

$$\therefore D = 2\int_{-\infty}^{0}\left(\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}\right)dx = 1$$

2006 PHUHS January Invitational Calculus Team Question 2

Note: B – A will be the rate at which the man is walking away from the street light which is 2 feet per second.

2006 PHUHS January Invitational Calculus Team Question 3

$V(x) = (x)(18-2x)(24-2x)$ has a max @ $x \approx 3.39$, $V(x) \approx 654.97733$ cubic inches.

A = 654

$21.66 = \pi r^2 h$, $SA = 2\pi\left(\frac{21.66}{\pi r}\right) + 2\pi r^2$, SA has a minimum @ $r \approx 1.5106356$, $SA \approx 43.015024...$

B = 44

The lowest common multiple of A and B = 14388

2006 PHUHS January Invitational Calculus Team Question 4

$$V = \frac{1}{3}\pi r^2 h,\quad \frac{r}{h} = \frac{4}{10} = \frac{2}{5} \to \frac{dr}{dt} = \frac{2}{5}\frac{dh}{dt},\quad \frac{dV}{dt} = \frac{\pi}{3}\left[2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right],$$

$$V = 50 \to r = \sqrt[3]{\frac{60}{\pi}}, h = \frac{5}{2}\sqrt[3]{\frac{60}{\pi}},$$

$$1 = \frac{\pi}{3}\left[2\sqrt[3]{\frac{60}{\pi}}\frac{5}{2}\sqrt[3]{\frac{60}{\pi}}\frac{2}{5}\frac{dh}{dt} + \sqrt[3]{\frac{60}{\pi}}^{2}\frac{dh}{dt}\right] \to \frac{dh}{dt} = \sqrt[3]{\frac{1}{3600\pi}} \approx .0445501539$$

$$A = \pi r^2 \to \frac{dA}{dt} = 2\pi r\frac{dr}{dt} = 2\pi\sqrt[3]{\frac{60}{\pi}}\frac{2}{5}\sqrt[3]{\frac{1}{3600\pi}} = \frac{4\pi}{5}\sqrt[3]{\frac{1}{60\pi^2}} \approx .2992881$$

$$\frac{dh}{dt}\times\frac{dA}{dt} = .01666666... = \frac{1}{75}$$

2006 PHUHS January Invitational Calculus Team Question 5

A = $\int_{-4}^{2}\sqrt{16-x^2}\,dx = \frac{1}{2}\times 2\times 2\sqrt{3}+\frac{1}{2}\times 4^2\times\frac{3\pi}{4}=2\sqrt{3}+6\pi$

B = $\int_{-2}^{-2+\sqrt{6}}\frac{x+2}{x^2+4x+6}\,dx = \frac{1}{2}\ln\left(x^2+4x+6\right)\Big|_{-2}^{-2+\sqrt{6}} = \frac{1}{2}\ln 8-\frac{1}{2}\ln 2=\frac{1}{2}\ln 4=\ln 2$

Find: $A\times e^B=\left(2\sqrt{3}+6\pi\right)e^{\ln 2}=4\sqrt{3}+12\pi$

2006 PHUHS January Invitational Calculus Team Question 6

A = area of one petal $\frac{1}{2}\int_0^{\frac{2\pi}{5}}\left(1-\cos\left(5\theta\right)\right)^2 d\theta\approx .9424777961$

B = area of the inside "petal" = $2\times\frac{1}{2}\int_0^{\frac{\pi}{3}}\left(2-4\cos\theta\right)^2 d\theta\approx 2.17406579$

C = area of an ellipse centered at the origin that passes through (5, 0) and the nearest focus to that point has coordinates (3, 0). $20\pi\approx 62.83185307$

Find: $\frac{A\times B}{C}$ correct to the nearest thousandth = $.03261098\ldots\approx .033$

2006 PHUHS January Invitational Calculus Team Question 7

$y=x^{x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}}$ can be written $y=x^y$ which can be written $x=y^{\frac{1}{y}}=\sqrt[y]{y}$

If $f\left(x\right)$ is a real valued function then x can be any number that can be written in the form $x=y^{\frac{1}{y}}$ the maximum value for x occurs when y is equal to e. For any value of y greater than e, the function diverges.

$$x=y^{\frac{1}{y}}\to\ln x=\frac{1}{y}\ln y\to\frac{x'}{x}=\frac{1}{y^2}-\frac{\ln y}{y^2}\to x'=\frac{x}{y^2}\left(1-\ln y\right)\to y=e$$

2006 PHUHS January Invitational Calculus Team Question 8

If P(x) is a continuous probability distribution function.

Then $\int P\left(x\right)=1\to\int_0^2 k\left(8x-x^3\right)dx=1\to k\left(4x^2-\frac{x^4}{4}\right)\Big|_0^2=1$

$\therefore k=\frac{1}{12}$

<u>2006 PHUHS January Invitational Calculus Team Question 9</u>

$$\text{P(two distinct pairs)} = \frac{\binom{13}{2}\times\binom{4}{2}\times\binom{4}{2}\times\binom{44}{1}}{\binom{52}{5}} = .0475390156\ldots$$

$$\text{P(Royal Flush)} = \frac{4}{\binom{52}{5}} = .000001539077169\ldots$$

.0475390156…/.000001539077169 =30888

<u>2006 PHUHS January Invitational Calculus Team Question 10</u>

$A = \frac{1}{2}a\times b\times\sin\theta \rightarrow \frac{dA}{dt} = \frac{1}{2}a\times b\times\cos\theta\times\frac{d\theta}{dt} = \frac{1}{2}\times 8\times 10\times\cos\frac{\pi}{6}\times\frac{\pi}{90} = \frac{2\pi\sqrt{3}}{9}$ square inches per second

<u>2006 PHUHS January Invitational Calculus Team Question 11</u>

A = $\lim\limits_{x\to-\infty}\left(\frac{x^2}{e^{-x}}\right) = 0$, B = $\lim\limits_{x\to 0}\left(\frac{1-\cos 4x}{4x}\right) = 0$, C = $\lim\limits_{x\to 2}\left(\cos\frac{\pi x}{3}\right) = \frac{-1}{2}$

D = $\lim\limits_{x\to 3}\left(\frac{x^2+4x-21}{x-3}\right) = 10$

Find: A + B + C + D = 9.5

<u>2006 PHUHS January Invitational Calculus Team Question 12</u>

$$\frac{d}{dx}\left[\pi^3x^2+2\pi x-\pi\right] = 2\pi^3x+2\pi$$