DAVID ESSNER FINALS 2004-2005

The point values for the problems are: (1) 20; (2) 17; (3) 18; (4) 25; (5) 20.

The use of a calculator is permitted only on problems 1(b1); 2(a), 2(c).. Graphic calculators are *not* permitted. In order to receive credit for numbers obtained by a calculator it is necessary that numerical expressions used to determine these values be displayed.

1. A Probability Problem

(a) On the real number line an object starts at the integer 0 and makes a sequence of moves, each move one unit to the right or left adjacent integer.

(a1) If for each move the probability the object moves to the right is 3/5 and to the left is 2/5, determine exactly the probability after that after 8 moves it is at the integer 4.

(a2) If for each move the probability the object moves to the left is 1/2 and to the right is 1/2, find exactly the probability that the object returns to 0 at least once during the first 6 moves.

(b) In the Cartesian plane an object starts at the origin (0,0) and makes a sequence of moves, each move having four possible equally likely directions: one unit to the left, one unit to the right, one unit up or one unit down.

(b1) If the object makes 6 moves, find to four decimal places the probability that the final position is (3,1).

(b2) If the object makes 4 moves, find exactly the probability that it reaches the position (1,1) at least once.

2. The Salary-Commission Problem

A man works under the following agreement. The first day he earns $\$S$ and at the end of the day pays $\$C$ commission. The next day he earns twice the net earning of the previous day and pays twice the commission of the previous day. (Net earning equals [earning - commission] and may be either a positive or negative value). This continues so that each day the man earns twice the net earning of the previous day and pays twice the commission of the previous day.

(a) If $S = 50$ and $C = 10$, determine the net earnings each of the first 5 days of work.

(b) For arbitrary values S and C and positive integer n, determine **with proof** a formula for the (i) commission on the n^{th} day of work.
(ii) net earning on the n^{th} day of work.

(c) Suppose $S = 100$, $C = 2$ and the net earning and commission are each increased by 5% (instead of doubled) each day. Find an integer N for which the net earning on day N of work is greater than 100 and on day $(N + 1)$ is less than 100. . Explain how you obtained your answer;

3. The Radius of the Inscribed Circle In a Right Triangle Problem

Given a right triangle whose legs have lengths a and b, find (with detailed proof) in terms of a and b the

(a) radius of the circle which passes through the three vertices of the triangle.

(b) radius of the inscribed circle of the triangle.

4. **The Square Terms In Arithmetic Sequences Problem**

Consider an arithmetic sequence of positive integers of the form

$$a, a + d, a + 2d, a + 3d, \ldots, a + nd, \ldots$$

where a, d, n are positive integers.

(a) Prove that if $a = 1$ then the square of each term in the sequence is also a term in the sequence.

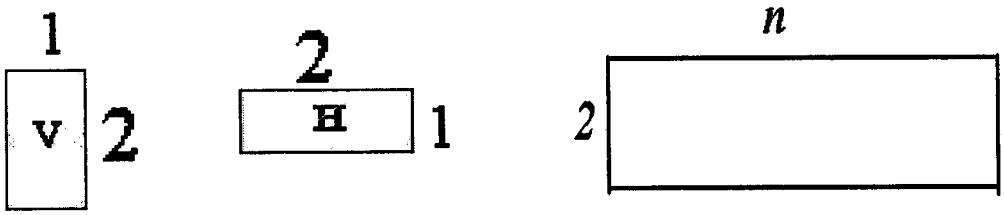
(b) Find, with proof, a necessary and sufficient condition relating a and d so that the square of each term in the sequence is also a term in the sequence.

(c) Prove that if a is a perfect square then for each positive integer d there is at least one value of n such that $a + nd$ is a perfect square; also show there is not a largest such value.

(d) Prove that the sequence $\{3 + 5n : n = 1,2,3,\ldots\}$ does not have a perfect square term.

5. A Tile Problem

A rectangular floor of size 2xn ($n \geq 1$) is to be covered with 1x2 tiles which can be placed either vertically or horizontally e.g. for n = 3 this can be done in the 3 ways (see the figures below):



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(a) For n = 4 show all the ways it can be done.

(b) For n = 12 determine the number of ways it can be done; justify your method (do not try to show all the ways as there are more than 100).

(c) For a rectangular 4x6 size floor determine the number of ways the spaces can be filled with 1x2 tiles placed either horizontally or vertically; justify your method (do not try to show all the ways as there are more than 100).