Problem Solving Strategy Essay # 3:

Engage in Wishful Thinking

James Tanton, PhD, Mathematics, Princeton 1994; MAA Mathematician in Residence

***

Teachers and schools can benefit from the chance to challenge students with interesting mathematical questions that are aligned with curriculum standards at all levels of difficulty.

This is the third essay in a series to give credence to this claim on the MAA website, www.maa.org/math-competitions. For over six decades, AMC has been creating and sharing marvelous stand-alone mathematical tidbits. Take them out of their competition coverings and see opportunity after opportunity to engage in great conversation with your students. Everyone can revel in the true creative mathematical experience!

I personally believe that the ultimate goal of the mathematics curriculum is to teach self-reliant thinking, critical questioning and the confidence to synthesize ideas and to re-evaluate them. Content, of course, is itself important, but content linked to thinking is the key. Our complex society is demanding of the next generation not only mastery of quantitative skills, but also the confidence to ask new questions, explore, wonder, flail, innovate and succeed. Welcome to these essays!

OUR CHALLENGE TODAY

Our tidbit today is **query 24** from the **2010 MAA AMC 8** competition.

What is the correct ordering of the three numbers 10^8, 5^{12} and 2^{24}?

(A) $2^{24} < 10^8 < 5^{12}$
(B) $2^{24} < 5^{12} < 10^8$
(C) $5^{12} < 2^{24} < 10^8$
(D) $10^8 < 5^{12} < 2^{24}$
(E) $10^8 < 2^{24} < 5^{12}$

How one might lead a class discussion on this challenge:

A first step to answering any question is to:

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

Having the entire class take a deep breath in unison - make it a show of it! - begins to dispel nerves and bring focus. (I find middle school students are always up for showy antics.)

STEP 2: Understand the question. Understand the different components of the question.

One can approach Step 2 leading a discussion of the following type:

What is the question about?

Numbers. [It never hurts to state the obvious! In fact doing so can be a helpful device for getting one's mind a-thinking.]

What about the numbers?

Something about a "correct ordering."

What does that mean?

Perhaps arrange them from smallest to largest, or largest to smallest, or something?

What numbers are we ordering?

Weird numbers.

Which weird numbers?

10^8, 5^{12} and 2^{24}.

What does that all mean?

And now we are at the crux of matters!

Your curriculum may have students interpret exponents, at least within the realm of the counting numbers, as "repeated multiplication." For example, 7^3 is interpreted as the product of three sevens, $7\times7\times7$, and 9^{103} as the product of 103 nines, $9\times9\times\cdots\times9$.

Thinking of exponentiation as repeated multiplication becomes problematic when trying to interpret quantities of the type 8^1, 2^0, 3^{-1} or $10^{1/2}$. (These appear in the grade 8 and high-school curricula.) If you are interested, view this short video www.jamestanton.com?/p=379 on how to deal with tricky exponents. Also see the bonus at the end of this essay.

Have your students come to see the three numbers as:

$$
\begin{aligned}
10^8 &= 10\times10\times10\times10\times10\times10\times10\times10\\
5^{12} &= 5\times5\times5\times5\times5\times5\times5\times5\times5\times5\times5\times5\\
2^{24} &= 2\times2\times2\times2\times2\times2\times2\times2\\
&\quad\times2\times2\times2\times2\times2\times2\times2\times2\\
&\quad\times2\times2\times2\times2\times2\times2\times2\times2
\end{aligned}
$$

STEP 3: Do something.

One thing students might try to do is follow the **BRUTE FORCE METHOD**: to actually work out the values of each of these products and see which is the largest and which is the smallest. Let them go for it!

$10^8 = 100,000,000$, a hundred million.

$$
\begin{aligned}
5^{12} &= 25\times5\times5\times5\times5\times5\times5\times5\times5\times5\times5\\
&=125\times5\times5\times5\times5\times5\times5\times5\times5\times5\\
&=625\times5\times5\times5\times5\times5\times5\times5\times5\\
&=3125\times5\times5\times5\times5\times5\times5\times5\\
&=15625\times5\times5\times5\times5\times5\times5\\
&=\ldots.
\end{aligned}
$$

They will no doubt come to the conclusion that this is hard! They will suspect that 2^{24} is also going to be very difficult to compute. The cry "There's got to be an easier way!" will begin to ring loud and clear.

Some students will suggest a qualitative assessment: 5 is smaller than 10 so surely a product of 5s is smaller than a product of 10s? A good query! But others might observe that we're multiply more fives together than we are tens. Will this change matters? Hmm.

Following false leads, making good attempts that prove not to be helpful, and making mistakes are all valuable and natural parts of problem solving - even for mathematicians! False leads can lead to new insights and inspire you to conceive of new approaches.

A piece of advice: If you are stuck on a problem and are in search of new inspiration, try …

STEP 4: WISHFUL THINKING.
If you could change the question, what change would make it ridiculously easy?

The three numbers 10^8, 5^{12} and 2^{24} are awkward. If they were all the same number the problem would be easy! (That's a good piece of wishful thinking!)

The challenge would also be easy if there were no exponents – we can certainly order 10, 5 and 2!

Actually, along those lines … If all the products were the same length, say for 10^8, 5^8 and 2^8, then they would be easy to order:

$$10^8 = 10\times10\times10\times10\times10\times10\times10\times10$$
is biggest,

$$5^8 = 5\times5\times5\times5\times5\times5\times5\times5$$
is next biggest, and

$$2^8 = 2\times2\times2\times2\times2\times2\times2\times2$$
is smallest.

Alternatively …If the base numbers were the same, say 5^8 and 5^{12} and 5^{24}, they again would be simple to order.

A class discussion, with your nudging (not forcing), can come to the same realizations.

So … Can we make some version of any of these wishes true?

The number
$$\begin{aligned}2^{24} &= 2\times2\times2\times2\times2\times2\times2\times2\\ &\times2\times2\times2\times2\times2\times2\times2\times2\\ &\times2\times2\times2\times2\times2\times2\times2\times2\end{aligned}$$
is the longest and most awkward product of all. Is there a way to make this easier? To make it look more like 5^{12} or 10^8?

At this point your students might have an epiphany. What if we group the terms in the product?

$$\begin{aligned}2^{24} &= (2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\\ &\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\\ &\times(2\times2)\times(2\times2)\times(2\times2)\times(2\times2)\end{aligned}$$

This is 4^{12}! And clearly 4^{12} is smaller than 5^{12}. Whoa!

Can we make 2^{24} look like 10^8? Yes. Try another grouping!

$$\begin{aligned}2^{24} &= (2\times2\times2)\times(2\times2\times2)\times(2\times2\\ &\times2)\times(2\times2\times2)\times(2\times2\times2)\times(2\\ &\times2\times2)\times(2\times2\times2)\times(2\times2\times2)\end{aligned}$$

This is 8^8, which is smaller than 10^8. Double whoa!

We have just deduced that 2^{24} is smaller than both 5^{12} and 10^8.The answer is thus either option (A) or (B).

Can we do a similar piece of cleverness to compare 5^{12} and 10^8?

$$\begin{aligned}10^8 &= 10\times10\times10\times10\times10\times10\times10\times10\\ 5^{12} &= 5\times5\times5\times5\times5\times5\times5\times5\times5\times5\times5\times5\end{aligned}$$

Regrouping terms doesn't seem helpful this time. (But do let students try it if they suggest it!)

At this point going back to the wishful thinking list might be helpful. If 5 and 10 were the same number(!), it would be easy to compare 10^8 and 5^{12}.

Are 5 and 10 in any way close to being the same? (Strange question!)

With this nudge students might well realize the fact that $10 = 2\times5$ could be relevant. The numbers 5 and 10 are at least related!

$$\begin{aligned}10^8 &= 2\times5\times2\times5\times2\times5\times2\times5\\ &\times2\times5\times2\times5\times2\times5\times2\times5\end{aligned}$$

In writing this, students might suggest reordering terms:

$$\begin{aligned}10^8 &= 2\times2\times2\times2\times2\times2\times2\times2\\ &\times5\times5\times5\times5\times5\times5\times5\times5\end{aligned}$$

How could we make this look like 5^{12}, a product of twelve fives. We have a product of eight twos and eight fives. Hmm.

Another epiphany! Group terms again and make this a product of twelve terms.

$$\begin{aligned}10^8 &= (2\times 2)\times(2\times 2)\times(2\times 2)\times(2\times 2)\\ &\quad\times 5\times 5\times 5\times 5\times 5\times 5\times 5\times 5\\ &= 4\times 4\times 4\times 4\times 5\times 5\times 5\times 5\times 5\times 5\times 5\times 5\end{aligned}$$

And this is clearly smaller than

$$5^{12} = 5\times 5\times 5\times 5\times 5\times 5\times 5\times 5\times 5\times 5\times 5\times 5$$

Triple whoa!

We have $2^{24} < 10^8 < 5^{12}$. This is option (A).

At this point it would be good to a) celebrate the class's mighty good success and then b) reiterate the problem solving procedure. Perhaps write on the board the four-step procedure outlined here, recapping along the way the parts of the class discussion that were relevant to each step. Give your students the opportunity to truly own this problem-solving experience and the mathematics they encountered.

COMMON CORE STANDARDS and PRACTICES:

This problem is connected to standards:

> **6-EE-1:** Write and evaluate numerical expressions involving whole-number exponents.
>
> **8-EE-1:** Work with radicals and integer exponents.

We are also right on the mark with the following practice standards:

> **MP1:** Make sense of problems and persevere in solving them.
> **MP2:** Reason abstractly and quantitatively.
> **MP3:** Construct viable arguments and critique the reasoning of others.
> **MP7:** Look for and make use of structure.

Curriculum Connections:

The eighth-grade Common Core State Standards suggest students be familiar with radical expressions: that $\sqrt{5}$, for instance, is a quantity which when squared gives 5, and $\sqrt[3]{10}$ is a quantity, which when cubed, gives 10. To have students practice these ideas, you could suggest solving the same problem now a different way:

2^{24} *is a product of twenty-four identical terms. Rewrite* 5^{12} *and* 10^8 *each as a product of twenty-four identical terms and then use what you see to compare the sizes of each of the three numbers.*

Here we are encouraging students to write:
$5^{12} = \left(\sqrt{5}\right)^{24}$ and $10^8 = \left(\sqrt[3]{10}\right)^{24}$.
(This is very tricky for students! They will need your help.)

Now to compare 2^{24}, $5^{12} = \left(\sqrt{5}\right)^{24}$ and $10^8 = \left(\sqrt[3]{10}\right)^{24}$, we need only compare the numbers 2, $\sqrt{5}$ and $\sqrt[3]{10}$. A calculator at this point does the trick for us. (Ahh.. But what do you enter to compute $\sqrt[3]{10}$? This is the chance to discuss why $10^{\frac{1}{3}}$ is $\sqrt[3]{10}$!

See the video www.jamestanton.com?/p=379 for help!)

If you have a class eager for serious thinking … *How could we compare the numbers* 2, $\sqrt{5}$ *and* $\sqrt[3]{10}$ *without the aid of a calculator?*

Some wonderful estimation techniques now come into play.

$\sqrt{5}$ is a number which, when squared, gives 5. This number must be bigger than 2 (since $2^2 = 4$), but smaller than 3 (since $3^2 = 9$).

$\sqrt[3]{10}$ is a number which, when cubed, gives 10. This number must also be bigger than 2 (since $2^3 = 8$) but smaller than 3 (since $3^3 = 27$). We see at the very least that 2 is the smallest of the three numbers 2, $\sqrt{5}$ and $\sqrt[3]{10}$. We need now compare $\sqrt{5}$ and $\sqrt[3]{10}$.

Both $\sqrt{5}$ and $\sqrt[3]{10}$ are between 2 and 3:

$$2 < \sqrt{5} < 3$$
$$2 < \sqrt[3]{10} < 3$$

which "matches" with:

$$4 < 5 < 9$$
$$8 < 10 < 27$$

In some intuitive sense, it looks like 10 is relatively closer to 8 in the second inequality than 5 is close to 4 in the first. I am going to guess that $\sqrt[3]{10}$ is closer to 2 than $\sqrt{5}$ is. Can we prove this? (Intuition is handy, but can be wrong!)

Which is bigger: $\sqrt{5}$ or $\sqrt[3]{10}$ *?*

Let's do some wishful thinking … I wish these weren't radicals! Can we make the wish true?

Yes. Let's raise each to some power and compare the powers instead. Let's compare the sizes of $\left(\sqrt{5}\right)^6$ and $\left(\sqrt[3]{10}\right)^6$.

(Why did I choose powers of six?)

Now:

$$\left(\sqrt{5}\right)^6 = 5 \times 5 \times 5 = 125$$

and

$$\left(\sqrt[3]{10}\right)^6 = 10 \times 10 = 100.$$

We have that $\left(\sqrt{5}\right)^6$ is bigger than $\left(\sqrt[3]{10}\right)^6$, and so $\sqrt{5}$ is bigger than $\sqrt[3]{10}$.

Now we see: $2 < \sqrt[3]{10} < \sqrt{5}$ and so, again, $2^{24} < 10^8 < 5^{12}$.

(That was indeed tough!)

A LITTLE MORE FUN FOR EVERYONE!

The previous discussion suggests practice questions you might want to devise for your class. For example:

Question: Notice that $\sqrt{50}$ is larger than $\sqrt{49} = 7$ and smaller than $\sqrt{64} = 8$. Thus $7 < \sqrt{50} < 8$.
(And it is probably closer to 7 than it is to 8.)

Between which two integers do each of the following radicals lie?

a) $\sqrt{30}$ b) $\sqrt{117}$ c) $\sqrt[3]{96}$

Question: Which is larger:

$\sqrt{1072}$ or $\sqrt{1511}$?

$\sqrt{8}$ or $\sqrt[3]{25}$?

$\sqrt[5]{3}$ or $\sqrt[1003]{1}$?

$\sqrt[4]{2}$ or $\sqrt[5]{3}$?

Question: Is $\sqrt{3}$ larger or smaller than 1.8?

Better yet… Have students make up their own textbook problems! Help them realize that book authors are not fully dictating their learning!

FOUR COMMON APPROACHES TO EXPLAINING EXPONENTS

– and the questions they invite!

The following appears in *Mathematical Thinking, Chapter 12*, a text for middle school students. It is available at www.jamestanton.com.

Approach 1: Many people say that 2^a means "multiply two by itself *a* times."

This makes sense for 2^3, for example:

$2^3 =$ multiply two by itself three times $= 2 \times 2 \times 2 = 8$

Does this way of thinking make sense for 2^1? What do you think? "Multiply 2 by itself one time." Seriously think about this.

Does this way of thinking make sense of 2^0, or is it of no help? "Multiply 2 by itself no times." How about for 2^{-1}? For $2^{1/2}$?

Approach 2: Some people say that it is better to think in terms of doubling the number 1. For example, 2^3 is interpreted as "1 doubled three times":

$$2^3 = 1 \times 2 \times 2 \times 2 = 8$$

Does this way of thinking give good meaning to 2^1? "Double the number 1 one time." How about to 2^0? To 2^{-1}? To $2^{1/2}$?

Approach 3: Some choose to believe in patterns. They notice:

$$\begin{aligned}
&\vdots \\
&16 = 2^4 \\
&8 = 2^3 \\
&4 = 2^2 \\
&2 = 2^1 \text{ ????} \\
&1 = 2^0 \text{ ???} \\
&\frac{1}{2} = 2^{-1} \text{ ???} \\
&\frac{1}{4} = 2^{-2} \text{ ???} \\
&\vdots
\end{aligned}$$

The list on the left is halving from step to step and the superscripts to the right are decreasing by one step by step.

This pattern suggests it would be nice to say that $2^1 = 2$, that $2^0 = 1$, and $2^{-1} = \frac{1}{2}$. What do you think of this? Should we agree with what the pattern suggests?

Does the pattern suggest what to say about $2^{1/2}$? Something half-way between $2^0 = 1$ and $2^1 = 2$?

Approach 4: Mr. Twinkletot likes to give a physical demonstration of the doubling numbers to his students. He takes a large piece of paper and folds it in half once. He says "Look: One fold gives two layers" and writes on the board:

$$2^1 = 2$$

He folds the paper in half a second time and says to the class "Two folds gives four layers" and writes on the board:

$$2^2 = 4$$

After a third fold he says: "Three folds gives eight layers" and writes:

$$2^3 = 8$$

Poindexter is clever and asks Mr. Twinkletot to unfold the paper and go back to the beginning. Poindexter says "Look: No folds gives one layer." Poindexter then goes up to the board and writes:

__________ . (Fill in the blank.)

Clarissa is even more clever. She says that peeling the paper apart into half layers is the opposite of folding and that 2^{-1} should be the result of doing the opposite of folding one time. She writes on the board:

$2^{-1} =$ __________. (Fill in the blank.)

Mr. Twinkletot is befuddled but delighted. He never thought of these ideas before. He then asks: "Can we make sense of $2^{1/2}$ using this folding idea?"

What do you think? Can we?

Comment: The answer can be "no." But it would be exciting if the answer were "yes."

BY THE WAY:

153,211 students took the 2010 AMC 8 competition and 17.19% of them selected option (A), the correct answer. Option (B) received 23.28% of the responses, (C) 17.40%, (D) 20.80% and (E) 11.06% of the responses. 10.11% of the students left the question blank.

Don't forget, this was question 24 on a 40-minute test. A lot of pressure and exhaustion! I wonder how these percentages would have changed if more time was offered to mull and reflect and play with ideas?

Curriculum Inspirations is brought to you by the Mathematical Association of America and the MAA American Mathematics Competitions.