Pre-Calculus

Felix Varela High School

Famat Regional Competition

February 4, 2006

1. For $f(x) = x^2 + 4x + 1$, over what interval of x does $f^{-1}(x)$ exist and have a real solution?
   A. $(-\infty, \infty)$ B. $[-3, \infty)$
   C. $(-3, \infty)$ D. $(3, \infty)$
   E. NOTA

2. What is the fundamental period of the function, $f(\theta) = 10\sec(\theta)\sin(\theta)$.
   A. π B. 2π
   C. 4π D. $\pi/2$
   E. NOTA

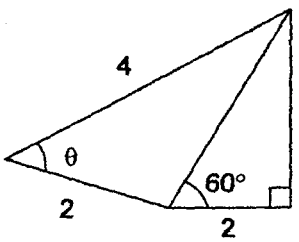
3. If the polar graph $r = \dfrac{3}{\sin(\theta) - 4\cos(\theta)}$ is placed on the Cartesian plane, what are the y-intercepts of the graph?
   A. -3, 3 B. 3
   C. $-3/4$ D. $-3/4, 3$
   E. NOTA

4. Determine the perpendicular bisector of the line segment joining the points (1,2) and (7,4).
   A. $x - 3y = -5$ B. $x + 3y = 13$
   C. $3x - y = 9$ D. $3x + y = 15$
   E. NOTA

5. Evaluate $\|2\mathbf{u} + 3\mathbf{v}\|$ with $\mathbf{u} = \langle 2,-1 \rangle$ and $\mathbf{v} = \langle 1,3 \rangle$:
   A. $7\sqrt{2}$
   B. 7
   C. 14
   D. $4\sqrt{10}$
   E. NOTA

6. Two forces, one of 100N and the other 150N act on an object. The angle between the forces is 60°. What is the magnitude, in Newtons, of the resultant vector sum (rounded to two decimal places)?
   A. 132.28
   B. 217.94
   C. 241.83
   D. 250.00
   E. NOTA

7. $\sin(\theta) = ?$



A. $\frac{1}{4}$
B. $\frac{3}{4}$
C. $\frac{\sqrt{15}}{4}$
D. $\frac{\sqrt{7}}{4}$
E. NOTA

8. A bag contains 5 defective items and 15 functional items. If three are chosen simultaneously at random, what is the probability that exactly 2 of the 3 items work?

A. $\frac{5}{38}$ B. $\frac{45}{91}$
C. $\frac{35}{76}$ D. $\frac{49}{57}$
E. NOTA

9. A triangle has vertices at (1,1), (4,5), and (5, 0). What is the area of the triangle?

A. 7.5 B. 8.5
C. 9.5 D. 17
E. NOTA

10. Which of the following accurately describes $f(\theta) = \tan(\theta)$ over the interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$?

A. It is monotonic.
B. It is periodic.
C. It is symmetric with respect to the y-axis.
D. It is bounded.
E. NOTA

11. Which of the following points is one of the two trisectors of the line segment between (2,-15) and (-7, 3)?

A. (-3, 6) B. (-4, -9)
C. (-4, -3) D. (5, 9)
E. NOTA

12. Line *l* follows the equation: $y = 2x+2$ and intersects the circle $x^2 + y^2 = 1$ at points A and B. Find the length of the line segment AB.

A. 1 B. $\sqrt{2}$
C. $\frac{\sqrt{3}}{2}$ D. $\frac{2\sqrt{5}}{5}$
E. NOTA

13. Which of the following is the directrix for the parabola, $\frac{x^2}{12}+4=y$?

A. $x=-3$
B. $y=-3$
C. $x=1$
D. $y=1$
E. NOTA

14. Find the partial fraction decomposition of $\frac{2x^2-5x-7}{x^2-4x+3}$

A. $\frac{5}{x-1}-\frac{2}{x-3}$
B. $2+\frac{13}{x-1}-\frac{26}{x-3}$
C. $2+\frac{5}{x-1}-\frac{2}{x-3}$
D. $\frac{5}{2(x-1)}-\frac{1}{x-3}$
E. NOTA

15. Solve for all values of x: $\ln(x+3)+\ln(x-4)=\ln(3x)$

A. -2,6
B. 6
C. -2
D. No real solutions
E. NOTA

16. Rationalize: $\frac{3+4i}{2+5i}$ where $i=\sqrt{-1}$

A. $\frac{-14+23i}{29}$
B. $\frac{26-7i}{29}$
C. $\frac{-14-7i}{29}$
D. $\frac{-26+7i}{21}$
E. NOTA

17. How many leaves does the graph of $r(\theta)=5\cos(4\theta)$ have?

A. 2
B. 4
C. 5
D. 8
E. NOTA

18. Simplify: $\frac{2x^4-7x^3-12x^2+28x+16}{x-2}$

A. $2x^3-3x^2-18x-8$
B. $2x^3-11x^2+10x+8$
C. $2x^3-11x^2-40x-16$
D. $2x^3+3x^2-6x+16$
E. NOTA

19. The set of parametric equations $x = 4\cos(t), y = 3\sin(t)$ where $0 \le t < 2\pi$ is best described by which of the following conic sections?

A. $x^2 + y^2 = 25$
B. $3x^2 + 4y^2 = 1$
C. $\frac{x^2}{16} - \frac{y^2}{9} = 1$
D. $\frac{x^2}{16} + \frac{y^2}{9} = 1$
E. NOTA

20. Solve over the range $0 < \theta \le 2\pi$:

$$\frac{\sin(2\theta)}{2} - \sin^2(\theta) = \cos(2\theta)$$

A. $\pi/4, \pi/2, 5\pi/4, 3\pi/2$
B. $\pi/4, \pi/2, 5\pi/4$
C. $\pi/4, \pi/2, 5\pi/4, 3\pi/2, 2\pi$
D. $\pi/4, 5\pi/4$
E. NOTA

21. $\sum_{x=1}^{\infty}\left[\left(\prod_{y=1}^{x} y\right)^{-1}\right] = ?$

A. e
B. π
C. $\pi/4 + 1$
D. e-1
E. NOTA

22. What distance is the point (1,2) from the line 3x+6y=10?

A. $\frac{\sqrt{5}}{3}$
B. $\frac{5\sqrt{5}}{3}$
C. 5
D. $\sqrt{5}$
D. NOTA

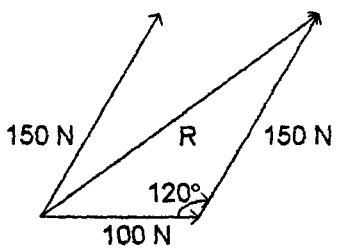
23. Which of the following is equivalent to $(xyz+1)(1-xy)+[1-(x+y)]$ under the rules of Boolean Algebra?

A. 1 – xy
B. 1
C. (1-x)(1-y)
D. z
E. NOTA

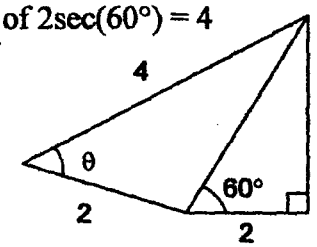
24. Which is of the following is NOT equivalent to cos(30°)?

A. sin(60°)
B. cot(-330°)×sin(-330°)
C. $\cos(-\pi/6)$
D. $-\cos(11\pi/6)$
E. NOTA

1. B. The interval over which $f^{-1}(x)$ exists is the same as the range of $f(x)$, or $[-3,\infty)$
2. A. $f(\theta) = 10\sec(\theta)\sin(\theta) = 10\sin(\theta)/\cos(\theta) = 10\tan(\theta)$, which has a period of π.
3. B. $r = \dfrac{3}{\sin(\theta) - 4\cos(\theta)} \Rightarrow y - 4x = 3$, so the y-intercept is 3.
4. D. The slope of the line segment is (4-2)/(7-1) = 1/3, so the slope perpendicular to that line is -3. The midpoint of the line segment is (4,3). The equation of the line with a slope of -3 that contains the point (4,3) is 3x + y = 15.
5. A. $\|2\mathbf{u}+3\mathbf{v}\| = \|2\langle 2,1\rangle + 3\langle 1,3\rangle\| = \|\langle 7,7\rangle\| = \sqrt{(7^2+7^2)} = 7\sqrt{2}$.
6. B. Placing the vectors head to tail combined with law of cosines produces R = 217.94.



7. C. The line segment common to both triangles has a length of 2sec(60°) = 4
   Law of cosines on the right triangle produces $\cos(\theta) = \sqrt{15}/4$



8. C. $\dfrac{{}_{15}C_2 * {}_5C_1}{{}_{20}C_3} = 35/76$
9. C. $\dfrac{1}{2} * \begin{vmatrix} 1 & 4 & 5 & 1 \\ 1 & 5 & 0 & 1 \end{vmatrix} = \dfrac{1}{2}|5+5-4-25| = \dfrac{19}{2} = 9.5$
10. A. $f(\theta)$ is monotonic, since it is always increasing between $-\pi/2 < \theta < \pi/2$. It is NOT periodic since it is only defined over $-\pi/2 < \theta < \pi/2$ and the function does not repeat itself inside that region. It is NOT symmetric to the y-axis since $f(\theta) \neq f(-\theta)$. It is NOT bounded since neither of the endpoints are included or defined in the region.
11. C. (-4,-3). -4 is two thirds of the way from 2 to -7, and -3 is two thirds of the way from -15 to 3.
12. D. The line and the circle intersect at points (-1,0) and (-0.6,0.8). The distance between these points is $\sqrt{(0.4^2+.08^2)} = (2\sqrt{5})/5$.
13. D. The parabola's vertex is (0,4) and the directrix is 12/4 under that, so $y = 1$.
14. C. $\dfrac{2x^2-5x-7}{x^2-4x+3} = 2 + \dfrac{3x-13}{x^2-4x+3} = 2 + \dfrac{-2}{x-3} + \dfrac{5}{x-1}$
15. B. $\ln(x+3) + \ln(x-4) = \ln[(x+3)(x-4)] = \ln(x^2-x-12) = \ln(3x)$, so $x^2-x-12 = 3x$ produces $x = 6,-2$. The coefficient of the ln() function cannot be negative, leaving only $x = 6$.
16. B. $\dfrac{(3+4i)(2-5i)}{(2+5i)(2-5i)} = \dfrac{6+8i-15i-20i^2}{4-25i^2} = \dfrac{26-7i}{29}$
17. D. A rose with an even coefficient, n, has $2n$ petals. 4*2=8.

18. A. $x-2\overline{)2x^4-7x^3-12x^2+28x+16}$ $\quad 2x^3-3x^2-18x-8$

19. D. The parametric equations form an ellipse which is centered at (0,0) and has axes lengths of 8 in the x direction and 6 in the y direction.

20. A. $\frac{\sin(2\theta)}{2}-\sin^2(\theta)=\cos(\theta)\sin(\theta)-\sin^2(\theta)=\cos(2\theta)=\cos^2(\theta)-\sin^2(\theta)$, so

$\cos(\theta)\sin(\theta)=\cos^2(\theta)\Rightarrow\cos(\theta)\sin(\theta)-\cos^2(\theta)=\cos(\theta)(\sin(\theta)-\cos(\theta))=0$

$\cos(\theta)=0$ and $\sin(\theta)=\cos(\theta)$ produce $\theta=\pi/4,\pi/2,5\pi/4,3\pi/2$

21. D. $\sum_{x=1}^{\infty}\left[\left(\prod_{y=1}^{x}y\right)^{-1}\right]=\sum_{x=1}^{\infty}\left[\frac{1}{x!}\right]$ which is the definition of e, without the first term.

22. A. $\frac{3(1)+6(2)-10}{\sqrt{3^2+6^2}}=\frac{\sqrt{5}}{3}$

23. A. Under Boolean Algebra only two values exist, 0 (zero) and 1 (non-zero). *xyz*+1 is always 1 (non-zero), and 1-(x+y) is 1 only under such cases as 1-xy is 1. Therefore the entire equation is going to have the same value as 1-xy.

24. D. $-\cos(11\pi/6)=-\cos(30°)\neq\cos(30°)$

25. B. The area enclosed is π times the half the lengths of the major and minor axes. $\pi*7*9=63\pi$.

26. D. $(1+i)^4=(\sqrt{2}*\text{cis}(45°))^4=4\text{cis}(180°)=-4$

27. C. Law of Sines: $8*\sin(70°)/\sin(55°)\approx 9.18$

28. E. The solution set is $\left\{e^{i\pi/16},e^{9i\pi/16},e^{17i\pi/16},e^{25i\pi/16}\right\}$. Note: none of the provided answer sets have 4 elements, so none can be the correct answer.

29. B. $15_6\times 31_6=11*19=209=545_6$

30. B. The product of the roots of an even-degree polynomial equals the last coefficient, divided by the first. -36/1 = -36.

1. Find the product of the positive slope of the asymptotes, the length of the transverse axis, and the length of a conjugate axis from the following equation:

$$\frac{x^2}{25}-\frac{2x}{5}-\frac{y^2}{144}=0$$

2. 16 baseball teams are playing in a standard double elimination tournament. If the champions of the tournament are undefeated, what is the total number of wins by all teams participating in the tournament? (There are no ties in baseball.)

3. If the polar graph $r=\sin(2\theta)+3\cos^2(\theta)+2\sin(\theta+\pi/4)$ is placed on the Cartesian plane, what are the y-intercepts of the graph?

4. $\mathbf{u}$ = <1, 1, 0>, $\mathbf{v}$ = <2, 0, 1>, and $\mathbf{w}$ = <0, 3, -1>.
$\mathbf{z_1}$ = <4, 3, 2>, $\mathbf{z_2}$ = <3, -3, 1> and $\mathbf{z_3}$ = <2, 2, 2>.
Given $u_i\mathbf{u}+v_i\mathbf{v}+w_i\mathbf{w}=\mathbf{z}_i$ for i = 1,2,3.

Find $\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}$:

5. Find the remainder from the following division: $\dfrac{3x^5+7x^4-18x^3+12x^2+x+2}{x^2-3x+2}$.

6. Circles A and B are tangent and centered at the points (3,0) and (0,4) respectively. If the common tangent line to A and B intersects the origin, what is the radius of the smallest of the two circles.

7. For the equation: $y=4\cos(2\theta+3)+1$
   A = the amplitude of the equation.
   B = the vertical shift.
   C = the horizontal shift to the right.
   D = the fundamental period.

   Find $\dfrac{A+B}{C*D}$ in exact terms.

8. Find the equation for the locus of points that satisfy the following condition:
   The sum of the distances to the points (0,0) and (6,0) is equal to 10.

9. A gadget production line has a rejection rate of 1/10. If 4 objects are taken from the production line, what is the probability that all of the gadgets will be rejected? Three are rejected? None are rejected? Find the sum of these three probabilities (to four decimal places).

10. XHS has 100 students taking statistics, chemistry or physics. 18 are taking only statistics, 17 are taking only chemistry and 23 are taking only physics. 12 are taking statistics and chemistry, but not physics. 10 are taking chemistry and physics, but not statistics. 11 are taking statistics and physics, but not chemistry. How many are taking all three?

11. Calculate $(B*C)^4$ with:

$$A = \sum_{n=1}^{\infty} L_n \text{ with } L_n = \frac{L_{n-1}}{2} \text{ and } L_1 = 1/4.$$

$$B = \sum_{n=1}^{\infty} L_n \text{ with } L_n = \frac{L_{n-1}}{n-1} \text{ and } L_1 = 1.$$

$$C = \sum_{n=1}^{24} L_n \text{ with } L_n = L_{n-1} + 2 \text{ and } L_1 = 1.$$

12. cis(0°)*cis(10°)*cis(20°)*cis(30°)* ... * cis(250°)*cis(260°)*cis(270°) = ?

13. Using the following series of numbers: {80, 98, 200, 245}
    A = the arithmetic mean
    B = the geometric mean
    C = the harmonic mean
    Find $\frac{AC}{B}$.

14. The GPS on a ship reports that the ship is going 40 knots in the north direction and 20 knots in the west direction. The magnetic compass reports that the ship has a heading of (is facing) 330°. What are the effect speeds from the perspective of the ship (i. e. forward or reverse and left or right along with magnitudes)? North is a heading of 360° and East is a heading of 90°.

15. Determine the acute angle formed by the intersection of the lines:
    $3x + 4y - 10 = 0$ and $5x - 12y + 8 = 0$.

1. 576. The positive slope of the asymptotes is b/a = 12/5. The length of the transverse axis is 2a = 10. The length of the conjugate axis is 2b = 24. The product of these = 576
2. 30. Fifteen teams must lose twice, and each loss requires a corresponding win.
3. 0, $\sqrt{2}$. The y-axis can be expressed in polar terms with $\theta = \pi/2$ and $\theta = 3\pi/2$. This means the y-intercepts are also the intersection of the graph with either $\theta = \pi/2$ or $\theta = 3\pi/2$. $r(\pi/2) = \sqrt{2}$ and $r(3\pi/2) = -\sqrt{2}$. However, $\theta = 3\pi/2$ corresponds with the negative part of the y-axis, that value must be multiplied by -1 to convert back into Cartesian coordinates: $-(-\sqrt{2}) = \sqrt{2}$, which is the same as the first intersection point. The other possibility is that the graph crosses the origin (r = 0). This does occur (as any graphing calculator will show) so y = 0 is also a y-intercept.
4. 20. $u_2 <1,1,0> + v_2 <2,0,1> + w_2 <0,3,-1> = <3,-3,1>$

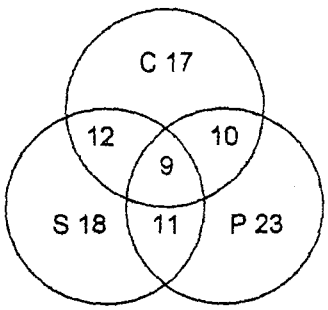
   $u_3 <1,1,0> + v_3 <2,0,1> + w_3 <0,3,-1> = <2,2,2>$

   so $u_1 = -6$, $v_1 = 5$, $w_1 = 3$, $u_2 = 9$, $v_2 = -3$, $w_2 = -4$, $u_3 = -10$, $v_3 = 6$, and $w_3 = 4$.

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} = \begin{vmatrix} -6 & 5 & 3 \\ 9 & -3 & -4 \\ -10 & 6 & 4 \end{vmatrix} = 20$$

5. 109x – 102. $\dfrac{3x^5+7x^4-18x^3+12x^2+x+2}{x^2-3x+2} = 3x^3+16x^2+24x+52+\dfrac{109x-102}{x^2-3x+2}$.
6. 9/5. The common tangent line must be perpendicular to both radii at the point of tangency. Since these lines have the same slope and a common point, they are the same line. This line will also contain the center of each circle (as part of the radii). Therefore the equation of the line is y = -4x/3 + 4 (containing (3,0) and (0,4)). The common tangent line is perpendicular to this line with a slope of 3/4 and contains the origin, producing the equation y = 3x/4. The intersection of the lines, (48/25, 36/25) is the point of tangency. The distance from this point to (3,0) is 9/5.
7. $-10/(3\pi)$. A = 4, B = 1, C = -3/2, D = π.
8. $\dfrac{(x-3)^2}{25}+\dfrac{y^2}{16}=1$. This is a definition of an ellipse with foci at (0,0) and (6,0). The focal length is 3, and the major axis is 10 along the x direction. The ellipse also has been shifted 3 to the right, and a minor axis of 8 This produces an equation of $\dfrac{(x-3)^2}{25}+\dfrac{y^2}{16}=1$
9. 0.6598. The probability that all are rejected is $(1/10)^4 = 0.0001$. The probability that three are rejected is $(1/10)^3 * (9/10) * {}_4C_3 = 0.0036$. The probability that none are rejected is $(9/10)^4 = 0.6561$. The sum = 0.0001 + 0.0036 + 0.6561 = 0.6598

10. 9. The student allocation has the following layout for the classes:



11. $24\sqrt{e}$. A = ½ (a geometric sequence). B = e (the sum of the inverse factorials). C = 576 (the sum of odds: 1 + 3 + ... + 45 + 47)

12. -1. $\prod_{x=0}^{27} cis(x*10^\circ) = cis\left(\sum_{x=0}^{27}(x*10^\circ)\right) = cis(3780^\circ) = cis(180^\circ) = -1.$

13. 140. A = (80 + 98 + 200 + 245) / 4 = 623/4. B = (80 * 98 * 200 * 245)^(1/4) = 140. C = 4 * (1/80 + 1/98 + 1/200 + 1/245) ^ -1 = 11200/89. A*C/B = 140.

14. 37.32 forward, and 24.64 to the left (or port). The ship has a track of $\tan^{-1}(-40/20)$ = 296.57° and a speed of $20\sqrt{(5)} = 44.72$. This can be broken into cos(330°-296.57°)*$20\sqrt{(5)}$ = 37.32 in direction of 330° and cos(296.57°-240°) = 24.64 in the direction of 240°

$3x+4y-10=0$ and $5x-12y+8=0$.

15. 59.49°. The line $3x+4y-10=0$ has an angle of $\tan^{-1}(4/3)$ = 53.13° from the x-axis and the line $5x-12y+8=0$ has an angle of $\tan^{-1}(-12/5)$ = -67.38° from the x-axis. Therefore the two lines have an angle of 120.51° between them, or an acute angle of 59.49°

Pre-Calculus Individual Solutions:

1. B
2. A
3. B
4. D
5. A
6. B
7. C
8. C
9. C
10. A
11. C
12. D
13. D
14. C
15. B
16. B
17. D
18. A
19. D
20. A
21. D
22. A
23. A
24. D
25. B
26. D
27. C
28. E
29. B
30. B

Pre-Calculus Team Solutions

1. 576
2. 30
3. $\sqrt{2}$
4. 20
5. 109x - 102
6. 9/5
7. $-10/(3\pi)$
8. $\frac{(x-3)^2}{25}+\frac{y^2}{16}=1$
9. 0.6598
10. 9
11. $24\sqrt{e}$
12. -1
13. 140
14. 37.32 forward and 24.64 to the left (or port)
15. 59.49°