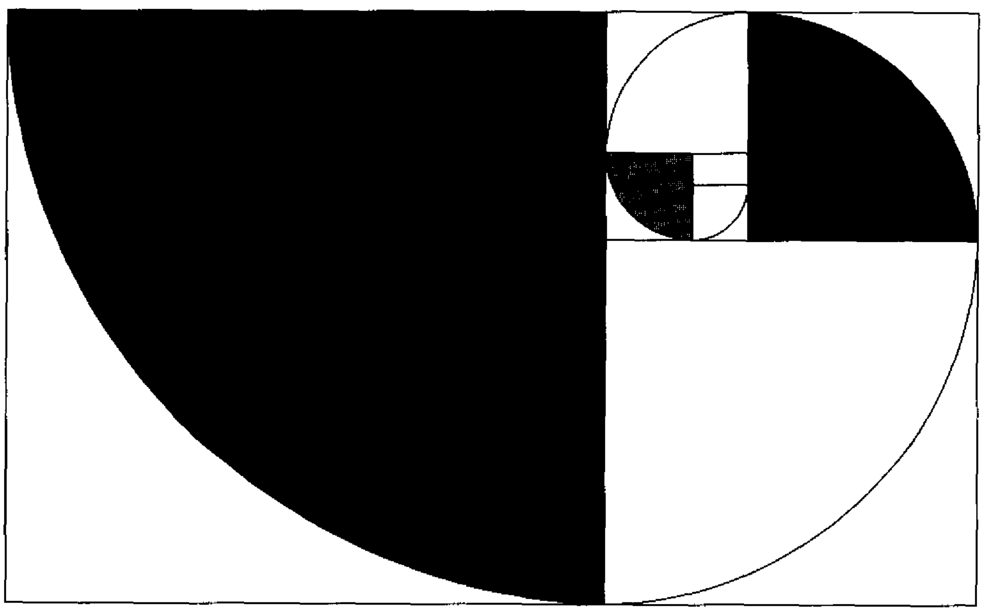
9

Similarity

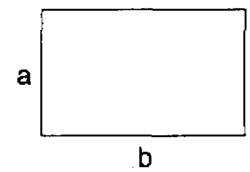


The Golden Rectangle

Name(s): ______________________

The golden ratio appears often in nature: in the proportions of a nautilus shell, for example, and in some proportions in our bodies and faces. A rectangle whose sides have the golden ratio is called a *golden rectangle.*

In a golden rectangle, the ratio of the sum of the sides to the long side is equal to the ratio of the long side to the short side. Golden rectangles are somehow pleasing to the eye, perhaps because they approximate the shape of our field of vision. For this reason, they're used often in architecture, especially the classical architecture of ancient Greece. In this activity, you'll construct a golden rectangle and find an approximation to the golden ratio. Then you'll see how smaller golden rectangles are found within a golden rectangle. Finally, you'll construct a golden spiral.

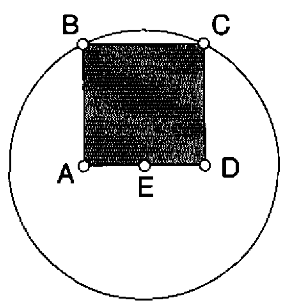


$$\frac{a+b}{b} = \frac{b}{a}$$

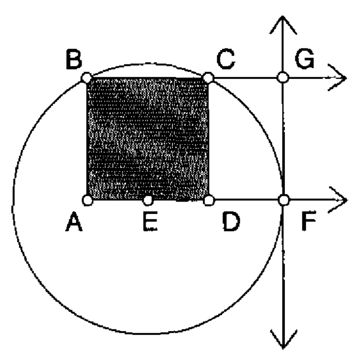
Sketch and Investigate

You can use the tool **4/Square (By Edge)** from the sketch **Polygons.gsp**.

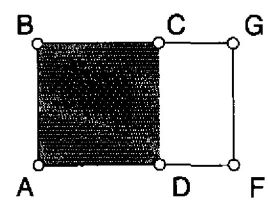
1. Use a custom tool to construct a square $ABCD$. Then construct the square's interior.
2. Orient the square so that the control points are on the left side, one above the other (points A and B in the figure).
3. Construct the midpoint E of $\overline{AD}$.
4. Construct circle EC.



Steps 1–4



Steps 5–8



Steps 9–11

Hold down the mouse button on the **Segment** tool to show the Straight Objects palette. Drag right to choose the **Ray** tool.

5. Extend sides AD and BC with rays, as shown.
6. Construct point F where $\overrightarrow{AD}$ intersects the circle.
7. Construct a line perpendicular to $\overrightarrow{AD}$ through point F.
8. Construct point G where this perpendicular intersects $\overrightarrow{BC}$. Rectangle $AFGB$ is a golden rectangle.

Select the objects; then, in the Display menu, choose **Hide Objects**.

9. Hide the lines, the rays, the circle, and point E.
10. Hide $\overline{AD}$, $\overline{DC}$, and $\overline{BC}$.

11. Construct $\overline{BG}$, $\overline{GF}$, and $\overline{FA}$.

12. Measure AB and AF.

*Select, in order, $\overline{AF}$ and $\overline{AB}$; then, in the Measure menu, choose **Ratio**.*

13. Measure the ratio of AF to AB.

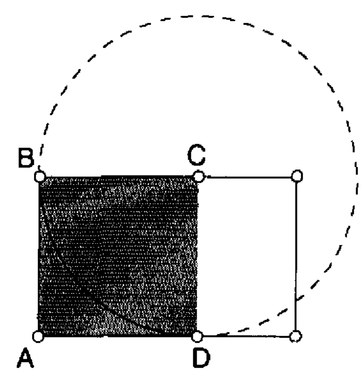
*Choose **Calculate** from the Measure menu to open the Calculator. Click once on a measurement to enter it into a calculation.*

14. Calculate $(AB + AF)/AF$.

15. Drag point A or point B to confirm that your rectangle is always golden.

Q1 The Greek letter phi (ϕ) is often used to represent the golden ratio. Write an approximation for ϕ.

Continue sketching to investigate the rectangle further and to construct a golden spiral.



16. Construct circle CB.

*Select, in order, the circle and points B and D. Then choose **Arc On Circle** from the Construct menu.*

17. Construct an arc on the circle from point B to point D, then hide the circle.

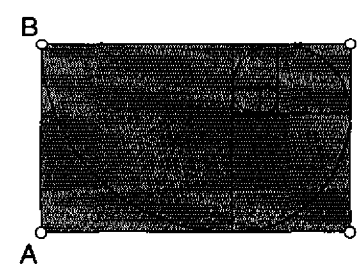
*Select the entire figure; then choose **Create New Tool** from the Custom Tools menu in the Toolbox (the bottom tool).*

18. Make a custom tool for this construction.

19. Make the rectangle as big as you can, then use the custom tool on points F and D. You should find that the rectangle constructed by your custom tool fits perfectly in the region $DFGC$.

If your rectangle goes the wrong way when you use the custom tool, undo and try applying it in the opposite order.

Q2 Make a conjecture about region $DFGC$.



20. Continue using the custom tool within your golden rectangle to create a golden spiral. Hide unnecessary points.

Explore More

1. Let the short side of a golden rectangle have length 1 and the long side have length ϕ. Write a proportion, cross-multiply, and use the quadratic formula to calculate an exact value for ϕ.

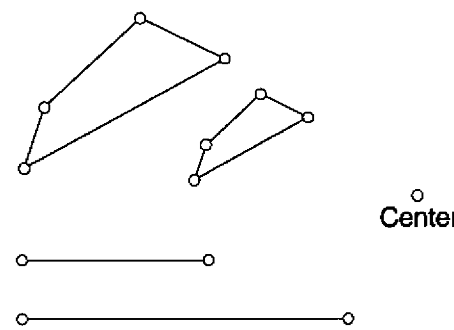
2. Calculate ϕ^2 and $1/\phi$. How are these numbers related to ϕ? Use algebra to demonstrate why these relationships hold.

Similar Polygons

Name(s): ______________________

Figures are *similar* if they have the same shape. Similar figures don't necessarily have the same size. A *dilation* is a transformation that preserves shape. In this activity, you'll use a dilation to discover principles of similarity. You'll use your discoveries to come up with a definition of similar polygons.

Sketch and Investigate



Double-click the point to mark it as a center.

Select the segments, then, in the Transform menu, choose **Mark Segment Ratio**. The selection order determines the numerator and denominator of the ratio.

Select the polygon; then, in the Transform menu, choose **Dilate**.

1. Construct any polygon.
2. Construct a point outside the polygon and mark it as a center for dilation.
3. Construct two segments of different lengths and mark them as a ratio.
4. Dilate your entire polygon by the marked ratio.
5. Drag the center of dilation. Also change the lengths of your two segments that define the ratio. Observe how these changes affect the similar polygons.

Q1 How can you make the dilated image coincide with the original figure?

6. If necessary, drag the ratio segments so that the polygons don't coincide.

Select the two segments; then, in the Measure menu, choose **Ratio**.

7. Measure the ratio of these segments' lengths.
8. Measure the ratios of some corresponding sides of your polygons.
9. Experiment with dragging different points on the polygon, the ratio segments, and the dilation center. Observe the ratios of corresponding side lengths in the polygon.
10. To compare angles in the two polygons, drag the center of dilation onto each vertex.

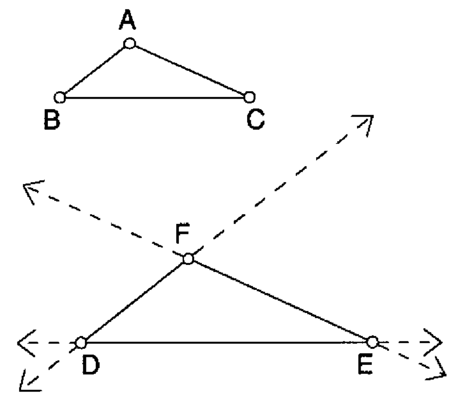
Q2 Use your observations to write a definition of similar polygons.

Similar Triangles—AA Similarity

Name(s): ________________

To show that two triangles are similar, you don't need to show that all the corresponding angles are congruent and that all the corresponding sides are proportional. There are several shortcuts. In this activity, you'll construct and investigate triangles with two pairs of congruent corresponding angles.

Sketch and Investigate



1. Construct $\triangle ABC$.
2. Construct $\overleftrightarrow{DE}$ below the triangle.
3. Mark point D as a center of rotation and mark $\angle CBA$ as an angle of rotation.

Double-click point D to mark it as a center. Select, in order, points C, B, and A; then, in the Transform menu, choose **Mark Angle**.

4. Rotate $\overleftrightarrow{DE}$ by the marked angle.

Select $\overleftrightarrow{DE}$; then, in the Transform menu, choose **Rotate**.

5. Drag point A and observe the effect on the angle formed by the rotated line.
6. Mark E as a center and mark $\angle BCA$.
7. Rotate $\overleftrightarrow{DE}$ about the new marked center by the new marked angle.
8. Construct point F, the point of intersection of these two rotated lines.
9. Hide the lines and replace them with segments.
10. Drag vertices of $\triangle ABC$ and observe the effects on $\triangle FDE$.

Q1 You constructed angles D and E to be congruent to angles B and C, respectively. Without measuring, how do you think angles A and F compare? Explain why they're related in this way.

Select two corresponding segments; then, in the Measure menu, choose **Ratio**. Repeat for the other two pairs of sides.

11. To see if the triangles are similar, measure the ratios of all three pairs of corresponding side lengths. Drag points and observe these ratios.

Q2 Make a conjecture about triangles with two pairs of congruent corresponding angles.

Explore More

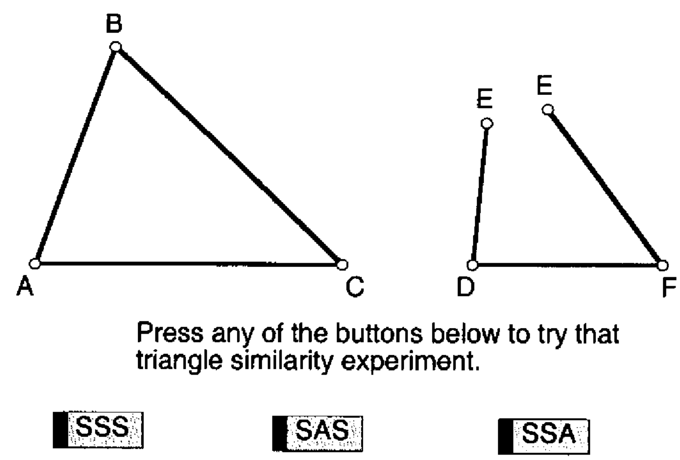
1. Investigate AA or AAA similarity for quadrilaterals.

Similar Triangles—SSS, SAS, SSA **Name(s):** ______________

You may have already discovered that if two angles in one triangle are congruent to two angles in another triangle, the triangles are similar. In this activity, you'll experiment with a sketch to discover other shortcuts for determining whether two triangles are similar. These shortcuts involve different combinations of proportional sides and congruent angles.

Sketch and Investigate

1. Open the sketch **Triangle Similarity.gsp**. You'll see a sketch like the one shown below.



2. Press the SSS action button and follow the instructions on the screen.

Q1 Write your conjecture below.

3. Experiment with the SAS button and the SSA button.

Q2 Of SSS, SAS, and SSA, which combinations of corresponding parts guarantee similarity in a pair of triangles? Which do not?

Explore More

1. Investigate similarity in quadrilaterals. What's the smallest amount of information you need to determine that quadrilaterals are similar?

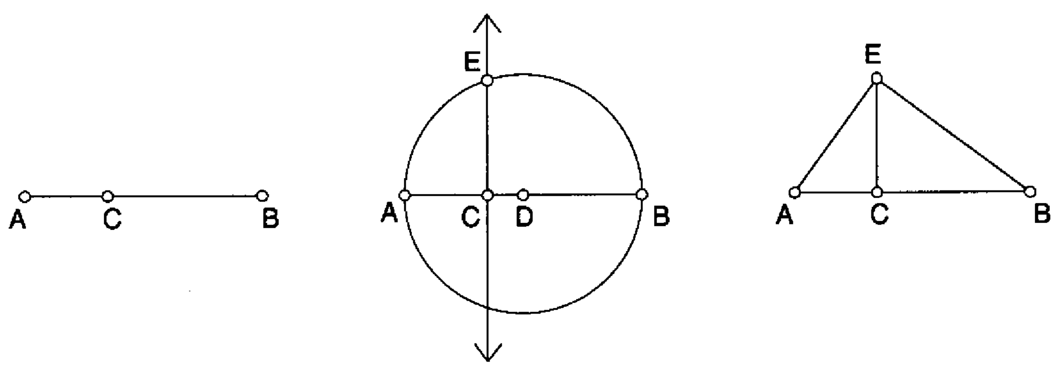
The Geometric Mean

Name(s): ______________________

What comes next in the number sequence 2, 6, 18, . . .? If you guessed 54, you noticed that each number in the sequence is just the previous term multiplied by 3. Or you may have noticed that there was a constant ratio between successive terms: $54/18 = 18/6 = 6/2 = 3$. A sequence with a constant ratio is called a geometric sequence, and each term is the *geometric mean* or *mean proportional* between the terms on either side of it. For example, 6 is the geometric mean of 2 and 18 because $2/6 = 6/18$. In this activity, you'll discover what a geometric mean is geometrically, and you'll learn how to construct a geometric mean between two lengths.

Sketch and Investigate

1. Construct $\overline{AB}$ and point C on $\overline{AB}$. You will construct the geometric mean of the lengths AC and CB.



Step 1 | Steps 2–5 | Steps 6–8

2. Construct the midpoint D of $\overline{AB}$.

Be sure to release the mouse button (or click the second time) with the cursor directly over point *A* so that the circle will be attached at this point.

3. Construct circle DA. Drag point A to make sure the circle is correctly attached.
4. Construct a line through C, perpendicular to $\overline{AB}$.
5. Construct point E where the line and the circle intersect.
6. Construct $\overline{AE}$ and $\overline{EB}$.
7. Hide the circle, the line, point D, and $\overline{AB}$.
8. Construct $\overline{AC}$, $\overline{CE}$, and $\overline{CB}$.

Q1 Drag point C. What kind of triangle is ΔABE? (*Hint:* Recall that it's inscribed in a semicircle.)

Q2 There are three similar triangles in your figure. Write the similarity relationships below and write an explanation of why the triangles are similar.

________ ~ ________ ~ ________

To measure a ratio, select two segments. Then, in the Measure menu, choose **Ratio**.

Q3 From your similar triangles, figure out which distance is the geometric mean of AC and CB. Measure some ratios to confirm your conjecture. Drag point C to confirm that the distance you found is always the geometric mean. Write a proportion below:

$$\frac{AC}{\quad} = \frac{\quad}{CB}$$

9. Measure CE, AC, and CB.
10. Calculate the product $AC \cdot CB$.

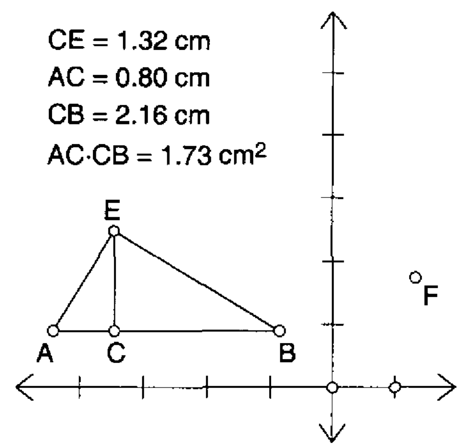
If you don't see the plotted point, drag the point at (1, 0) to scale the axes.

11. Select the measurement CE and the product $AC \cdot CB$. Then, in the Graph menu, choose **Plot As (x, y)**.

Select the plotted point and point C. Then, in the Construct menu, choose **Locus**.

12. Construct the locus of this plotted point and point C.

Q4 Describe the graph. Explain how it is related to the proportion you wrote in Q3.



Explore More

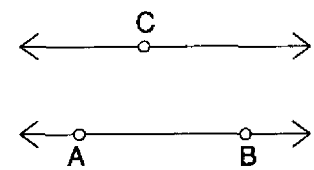
1. There are two other geometric means in your triangle. Find them and state what they are.
2. Let AE be a, let EB be b, let AC be x, and let CB be $c - x$ (so that AB is c). Use similar triangles to write two proportions. Cross-multiply in each proportion, then add the two resulting equations to combine them into one. Show that $a^2 + b^2 = c^2$ (the Pythagorean theorem).
3. See if you can find a geometric mean in a regular pentagram (five-pointed star).

Finding the Width of a River

Name(s): ____________________

Similar triangles have many problem-solving applications. In this activity, you'll use similar triangles to find a distance that can't be measured directly.

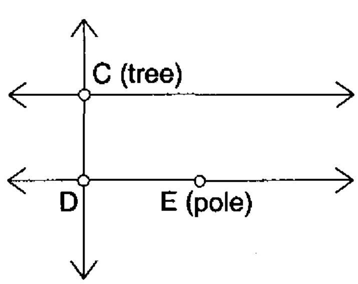
Sketch and Investigate



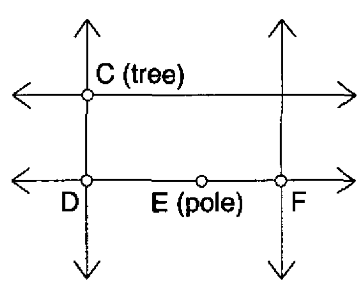
1. Construct $\overleftrightarrow{AB}$ and a point C above $\overleftrightarrow{AB}$.
2. Construct a line parallel to $\overleftrightarrow{AB}$ through point C. Imagine that these lines are the banks of a river. You wish to find the distance across the river. (And you can't just measure it—it's a river. So stay away from that Measure menu!)
3. Hide points A and B.

Select point C and either line; then, in the Construct menu, choose **Perpendicular Line**.

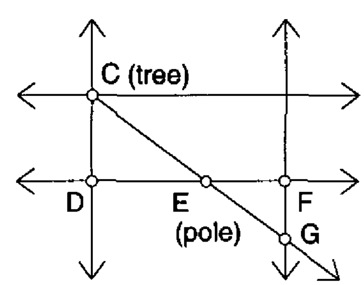
4. Point C represents a tree on the other side of the river from you. To locate yourself directly across the river from point C, construct a line through point C, perpendicular to the bank of the river.
5. Construct point D, where the perpendicular line intersects the other bank. Point D is where you start.
6. You walk a few meters and stick a pole into the bank at point E. (Construct $\overline{DE}$, where point E is on the bank.)



Steps 3–6



Steps 7 and 8



Step 9

7. You walk some convenient distance farther. (Construct $\overline{EF}$, where point F is on the bank.)
8. Then you make a right turn. (Construct a line through point F perpendicular to the bank.)
9. You want to line up the pole you stuck into the ground with the tree on the opposite bank. (Construct $\overrightarrow{CE}$ and point G at the intersection of $\overrightarrow{CE}$ and the line through point F.)
10. Drag point C, E, or F to see how the model changes. You can also change the width of the river by dragging either line.

Q1 Name two similar triangles in the model and explain how you know they're similar.

Q2 *CD* is the distance you wish to find, and you could measure *DE*, *EF*, and *FG*. But before you measure, write a proportion in terms of these four lengths.

Q3 Solve the proportion above for *CD*.

You can select the segments and measure lengths, or for each distance you can select two points, then, in the Measure menu, choose **Distance**.

11. Now measure *DE*, *EF*, and *FG*.

Choose **Calculate** from the Measure menu to open the Calculator. Click once on a measurement to enter it into a calculation.

12. Calculate an expression that you think will be equal to *CD* using these three measurements.
13. Measure *CD* to confirm your calculation. You can move the banks of the river or the locations of various points to confirm that this method will work under different circumstances.

Q4 What would be the most convenient way to locate points *E* and *F* in order to find *CD* without measuring it directly? Explain.

Explore More

1. Come up with another way to model this problem with Sketchpad using similar triangles.

Finding the Height of a Tree

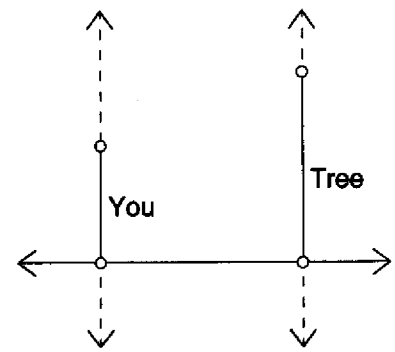
Name(s): ____________________

You can use similar triangles to calculate the heights of objects you are unable to measure directly. You can find the height of a tall object outdoors by measuring shadows on a sunny day. In Explore More, you will learn a method for finding the height of a tall object that works even when it's cloudy.

Sketch and Investigate

If you hold down the Shift key while you draw, it will be easier to make the line horizontal.

1. Draw a horizontal line. This will represent the ground.
2. Hide the control points of the line.
3. Construct two points on the line. These will represent your feet and the bottom of the tree.



Steps 1–6

To construct the perpendicular lines, first select the points and the ground line.

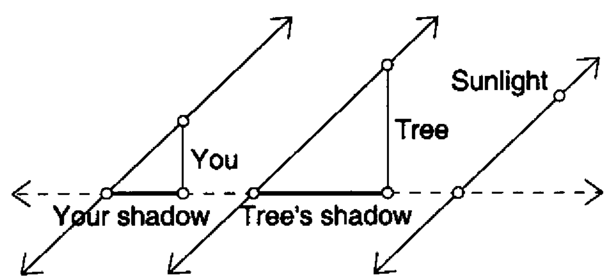
4. Construct lines through these points perpendicular to the ground.
5. Construct segments on these lines that start at the ground and go up. Hide the lines. (The lines to hide are shown as dashed in the figure above.)

Using the **Text** tool, click once on a segment to show its label. Double-click the label to change it.

6. Label the two perpendicular segments. Change the labels so that one segment is named after you and the other is called *Tree*. Adjust the heights so that the tree is much taller than you.

You shouldn't think of this line's control point as representing the sun. (The sun is much too big and much too far away to fit into your sketch!) You can, however, control the direction from which the sun shines with this point.

7. Draw a line from the sky above and to the right of the tree, down to the ground. This line represents a ray of sunlight.



Steps 7–10

8. Since the sun is so far away and so large, when its rays reach the earth, they are essentially parallel. Construct lines parallel to the line of sunlight through the point at the top of *You* and through the point at the top of *Tree*.
9. Construct the points where the parallel lines intersect the ground, then hide the ground line.

The **Line Width** submenu is in the Display menu.

10. Construct segments to represent your shadow and the tree's shadow. Change the line width of these shadows to thick and label them to represent their objects.

11. Measure the three things that you would actually be able to measure directly in the real situation: your height, the length of your shadow, and the length of the tree's shadow.

12. Drag different points in your model to see what they do. Think about what these movements represent in the physical situation (for example, the sun going up or down, shadows getting longer, you getting taller, and so on). Be careful—it's possible to represent situations with your model that are physically impossible!

You might want to animate a sunset and sunrise.

Q1 What happens to shadows as the sun gets lower in the sky?

Q2 What happens to the length of your shadow if you make yourself shorter in your model?

Q3 Drag your "feet" along the ground to simulate walking. Does walking the short distances represented in this model do anything to the length of your shadow? Explain.

13. Your model is a scale drawing you can use to solve problems. Drag the top point of the segment that represents you until its length corresponds conveniently with your height (for example, if you're 60 inches tall, you could make the segment .60 cm or inches long.) Drag your tree to a good height (taller than you!) and adjust the sun's rays to a reasonable angle.

Q4 Explain why the triangles formed by you and your shadow and by the tree and its shadow are similar.

Q5 Using *only the three measurements you have already made,* figure out the height of the tree. Show and explain all your work.

14. You can check your reasoning in Q5 by measuring the height of the tree in your model. Find this length.

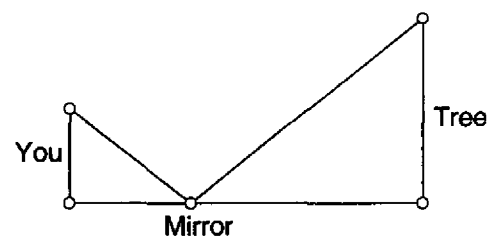
Q6 How well do your calculations match the measurement for the tree's height in your sketch? If your calculations don't match the measurement, explain what went wrong and try the calculations again.

Explore More

1. Drag parts of your model until it represents a situation that is physically impossible. Describe the situation.

2. Suppose you measured the shadows of two objects, each at a different time of day. Is it possible for the objects and their shadows to create similar triangles?

3. It's possible to make indirect measurements using similar triangles even on a cloudy day, when no shadows are cast. The method pictured at right uses mirrors. Use Sketchpad to model this method. (*Hint:* Since the method involves reflection in a mirror, your sketch will require a reflection, too.) Explain your method and why it works.



Measuring Height with a Mirror

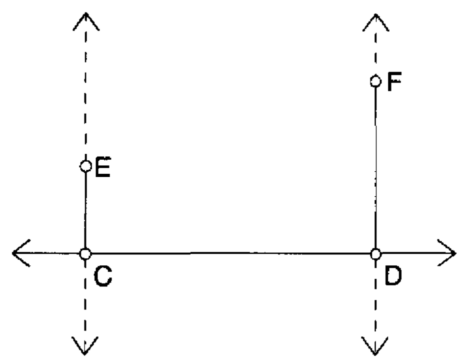
Name(s): ____________________

You can measure the height of a tall object such as a flagpole by putting a mirror on flat ground then locating yourself so that you can see the top of the flagpole in the mirror. Measure the height to your eye, the distance from your feet to the mirror, and the distance from the mirror to the base of the flagpole. Use these measurements and similar triangles to calculate the height of the flagpole. You might go outside and try it first. Then you can model the situation using Sketchpad.

Sketch and Investigate

If you hold down the Shift key while you draw, it will be easier to make the line horizontal.

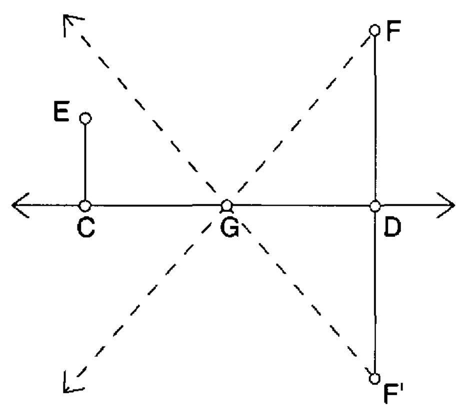
1. Draw a horizontal line AB. This will represent the ground.
2. Hide points A and B.
3. Construct points C and D on the line. These will represent your feet and the bottom of the flagpole.



Steps 1–5

To construct the perpendicular lines, first select the points and the ground line.

4. Construct lines through these points perpendicular to the ground.
5. Construct segments CE and DF on these lines, starting from the ground and going up. Hide the lines. (The lines to hide are shown as dashed in the figure above.)
6. Construct point G on the horizontal line between you and the flagpole. This point represents the place where you locate a mirror.
7. Construct $\overrightarrow{FG}$. This represents the path of a ray of light from the top of the flagpole to the mirror.



Steps 6–8

Double-click the line to mark it as a mirror. Select point F, the flagpole segment, and the ray; then, in the Transform menu, choose **Reflect**.

8. Mark the horizontal line as a mirror and reflect point F, the flagpole, and the ray over it.

The reflected ray $F'G$ represents the apparent path of a ray of light from the flagpole's mirror image through the mirror. From point G on, the ray represents the path of a ray of light from point F as it is reflected off the mirror.

9. You want to position yourself so that you can see the top of the flagpole in the mirror. Drag point C (your feet) until $\overrightarrow{F'G}$ passes through point E (your eye).

Q1 Explain why $\angle FGD \cong \angle EGC$.

Q2 Name the two similar triangles and explain why they're similar.

10. Experiment by moving the mirror, then finding out where to locate yourself to see the top of the flagpole given the new mirror location.

Q3 If you move the mirror closer to the flagpole, do you have to move closer to or farther from the mirror?

Q4 Remember, in the actual experiment you can't measure *FE* because the flagpole is too tall. What three distances could you measure easily (with the help of a partner)?

Q5 Before you measure, set up a proportion to find the height of the flagpole, *DF*, in terms of three other distances.

$$\frac{DF}{\quad} = \frac{\quad}{\quad}$$

Select points *E* and *C*; then, in the Measure menu, choose **Distance**. Repeat for *CG* and *GE*. Choose **Calculate** from the Measure menu to open the Calculator. Click once on a measurement to enter it into a calculation.

11. Now measure *EC*, *CG*, and *GD*. Calculate an expression that you think will be equal to *DF*. Measure *DF* to confirm your calculation. If you're off just a bit, it's probably because *E* can't be located exactly on $\overrightarrow{FG}$. You can relocate the mirror and try again from different positions. You can even change your height or the flagpole's height to change the problem.

Q6 Would you get a different height for the flagpole if you located the mirror in a different place? Explain why or why not.

Explore More

1. See if you can come up with other ways to model this problem with Sketchpad using similar triangles. (*Hint:* If you use shadows, you can solve the problem without any mirrors.)

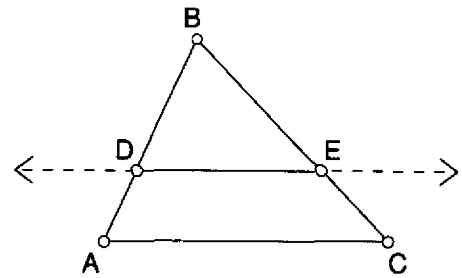
Parallel Lines in a Triangle

Name(s): ____________________

When you cut through a triangle with a line parallel to a side, you create similar triangles. In this activity, you'll investigate proportions that result from these similar triangles.

Sketch and Investigate

1. Construct $\triangle ABC$.
2. Construct point *D* on $\overline{AB}$.
3. Construct a line through point *D*, parallel to $\overline{AC}$.
4. Construct point *E* where the line intersects $\overline{BC}$.



Select point *D* and $\overline{AC}$; then, in the Construct menu, choose **Parallel Line**.

5. Hide the line and construct $\overline{DE}$.
6. Drag point *D* and observe the segment inside the triangle.

Q1 Name a pair of similar triangles and explain why they're similar.

Q2 Write three equal ratios involving the sides of the triangles.

$$\underline{\quad\quad} = \underline{\quad\quad} = \underline{\quad\quad}$$

Select a segment to measure a length. Or select two points and measure the distance between them.

7. Measure *BD*, *DA*, *BE*, and *EC*.
8. Drag point *D* again and observe these measurements.

Choose **Calculate** from the Measure menu to open the Calculator. Click once on a measurement to enter it into a calculation.

Q3 There's another proportion involving the segments into which the parallel line divides the sides of the triangle. Calculate ratios to discover a proportion involving *BD*, *DA*, *BE*, and *EC*. Write this proportion below.

$$\underline{\quad\quad} = \underline{\quad\quad}$$

Explore More

1. Show how the proportions involving *BD*, *DA*, *BE*, and *EC* can be derived using algebra and the similar triangles' proportions.
2. Show that the converse of your conjecture is true. That is, show that if a line divides the sides of a triangle proportionally, it's parallel to the third side.

Dividing a Segment into Equal Parts

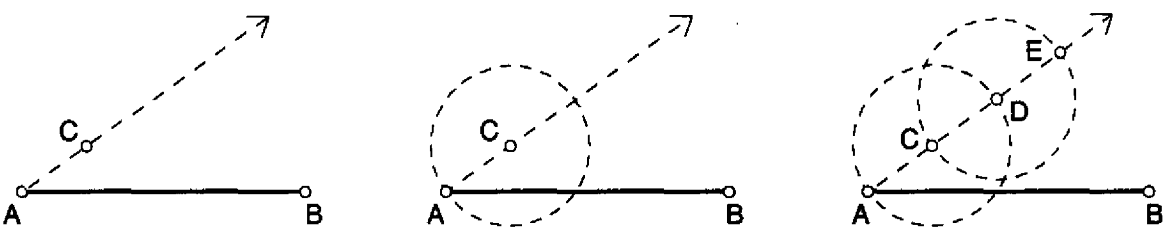
Name(s): ____________

You may already know how to construct a midpoint using a compass and straightedge. In this activity, you'll learn a method for dividing a segment into three or more equal parts using Sketchpad's tools and the Construct menu. This method is equivalent to the compass-and-straightedge method that Euclid describes in the *Elements*.

Sketch and Investigate

Hold the mouse button down on the **Segment** tool and drag right to switch to the **Ray** tool.

1. Construct $\overline{AB}$. This is the segment that you'll divide into three parts.
2. Construct $\overrightarrow{AC}$ at an angle from $\overline{AB}$. Locate point C no more than an inch from point A.



Steps 1 and 2 | Step 3 | Steps 4 and 5

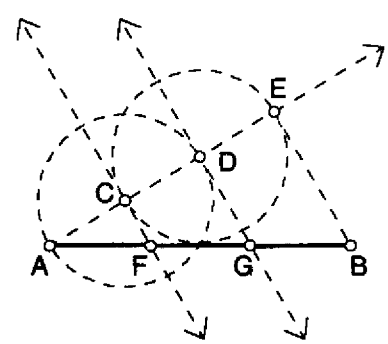
Be sure to release the mouse button (or click the second time) with the cursor directly over point *A* so that the circle will be properly attached.

3. Construct circle CA.
4. Construct circle DC, where point D is the intersection of the first circle with the ray.
5. Construct point E at the intersection of circle DC and the ray.

Q1 You've now marked off three equal distances on $\overrightarrow{AC}$. Name the equal distances and explain why they are equal.

Next, you'll use the three equal distances on $\overrightarrow{AC}$ to construct equal distances on $\overline{AB}$.

Select point *D*, point *C*, and $\overline{EB}$; then, in the Construct menu, choose **Parallel Lines**.



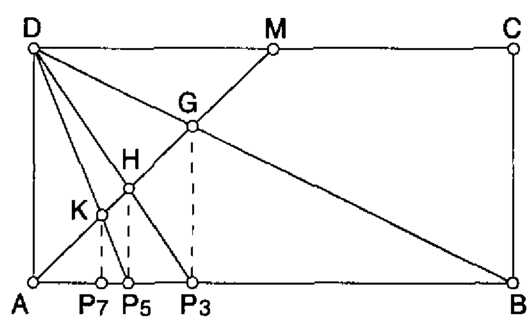
6. Construct $\overline{EB}$.
7. Construct lines through points D and C, parallel to $\overline{EB}$.
8. Construct points F and G where these lines intersect $\overline{AB}$.
9. Drag point A or point B to observe that points F and G always divide $\overline{AB}$ into equal thirds. (Measure if you wish to confirm your observation.)

Q2 Using similar triangles, explain why $AF = FG = GB$.

10. Hide everything but $\overline{AB}$ and points A, F, G, and B. Drag point A or point B to enjoy your trisection without having the rest of the construction block your view.

Explore More

1. Use this procedure to divide a segment into five equal parts.
2. Tenth-grade students Dan Litchfield and Dave Goldenheim used Sketchpad to discover a different method for dividing a segment. With the help of their teacher, Charles Dietrich, they published their findings in *The Mathematics Teacher* (vol. 90, no. 1, January 1997, pp. 8–12) in an article titled "Euclid, Fibonacci, GLaD." (They called their construction the GLaD construction, for Goldenheim, Litchfield, and Dietrich.) Part of their construction is shown below. $ABCD$ is a rectangle. Point M is a midpoint. Points P_3, P_5, and P_7 divide $\overline{AB}$ at 1/3, 1/5, and 1/7 of its length. Try the GLaD construction and confirm that it works. Add to it to show fractions with even denominators, such as 1/2, 1/4, and so on.



3. See if you can discover other methods for dividing a segment into equal parts.

Spacing Poles in Perspective

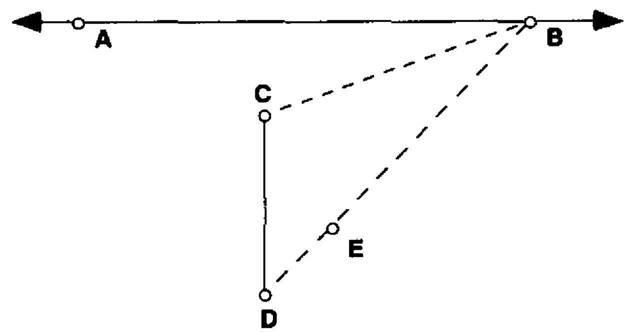
Name(s): ____________________

When you look at a long line of telephone poles or fence posts disappearing into the distance, the farther away the poles are, the shorter they appear to be. Distant poles also appear to be closer together. In this activity, you'll learn to draw fence posts in perspective so that their spacing appears realistic.

Sketch and Investigate

Hold down the mouse button on the current **Straightedge** tool and drag right to switch between the **Segment** and **Line** tools. Holding down the Shift key while you draw makes it easier to create horizontal and vertical lines.

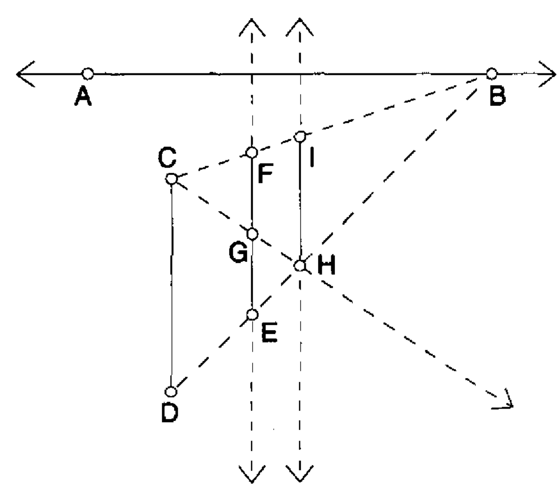
1. Construct a horizontal line $\overleftrightarrow{AB}$. This is your horizon line. Point B will be your vanishing point.



2. Construct vertical segment $\overline{CD}$. This is your first fence post (or telephone pole).
3. Construct $\overline{CB}$ and $\overline{DB}$.
4. Construct point E on $\overline{DB}$. This is the foot of your second fence post.

Select point E and $\overline{CD}$; then, in the Construct menu, choose **Parallel Line**.

5. Construct a line through point E parallel to $\overline{CD}$.
6. Construct point F where this line intersects $\overline{CB}$.
7. Hide line EF.
8. Construct $\overline{EF}$.
9. Construct point G, the midpoint of $\overline{EF}$.
10. Construct $\overrightarrow{CG}$.
11. Construct point H where $\overrightarrow{CG}$ intersects $\overline{DB}$. Point H is the foot of your third fence post.
12. Hide $\overrightarrow{CG}$ and point G.



13. Construct the third fence post just as you constructed the second in steps 5–8.

You can create a custom tool to construct all the rest of your fence posts.

Select point E; then, in the Edit menu, choose **Split Point From Segment**.

14. Split point E from $\overline{DB}$.
15. Select, in order, points B, D, C, E, H, and I and segment HI. (The first four selections will be the *givens* of the new tool; the last three will be the *results*.)

Exploring Geometry with The Geometer's Sketchpad

16. In the Custom Tools menu, choose **Create New Tool**. (The Custom Tools menu is the bottom tool in the Toolbox.) Name the tool **Next Pole**.

Select point *E* and $\overline{DB}$; then, in the Edit menu, choose **Merge Point To Segment**.

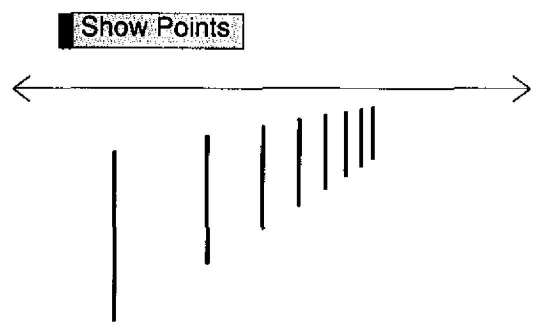
17. Merge point *E* back onto $\overline{DB}$.

18. Click on the **Custom** tools icon to choose your new tool. Now click, in order, on points *B*, *E*, *F*, and *H*. The next fence post should be constructed.

19. Construct at least five more fence posts in the same way. In each case, click on point *B* first, then on the lower endpoint of the second-to-last fence post, then on the upper endpoint of the second-to-last fence post, then on the lower endpoint of the last fence post.

20. Hide $\overline{CB}$ and $\overline{DB}$.

Click in a blank area to make sure nothing is selected, then choose the **Point** tool. In the Edit menu, choose **Select All Points**. Then choose **Edit | Action Button | Hide/Show**.



21. Select all the points, then make a Hide/Show action button. Press the button to hide the points.

22. Your static sketch may not look like much, but drag the first post to see the effect on your sketch. Can you imagine you're a bird swooping down on some abandoned telephone poles in the desert? Try lifting the poles so that their bases are above the horizon line.

23. Press the Hide/Show button to show the points again.

24. Experiment with dragging points to change your sketch. For example, drag point *E* to change the spacing between posts. Drag point *C* or point *D* to change the heights of the posts.

25. To see why this method gives the correct spacing, construct a segment from the top of the first post to the bottom of the third post. Construct another segment from the bottom of the first post to the top of the third post.

Q1 What do you notice about the place where the second post is located? Explain why this spacing appears realistic.

Proportions with an Angle Bisector in a Triangle

Name(s): ____________________

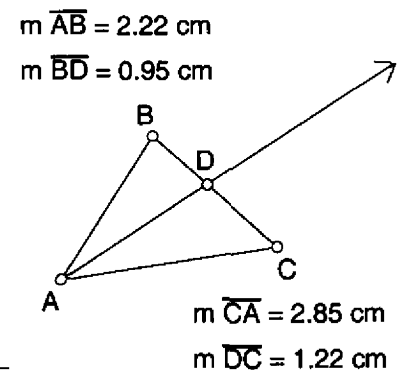
Quick! Where does an angle bisector intersect the opposite side in a triangle? Did you guess the midpoint? A quick check will show that this works only in special cases. In this activity, you'll discover a proportion relationship of angle bisectors in triangles.

Sketch and Investigate

1. Construct $\triangle ABC$.

Select, in order, points *B*, *A*, and *C*. Then, in the Construct menu, choose **Angle Bisector**.

2. Construct the bisector of $\angle BAC$.
3. Construct point D where the bisector intersects $\overline{BC}$.



4. Drag vertices of the triangle and observe point D.

Q1 Is point D the midpoint of $\overline{BC}$? __________

5. Hide $\overline{BC}$ and construct $\overline{BD}$ and $\overline{DC}$.
6. Measure AB, BD, DC, and AC.

Q2 Drag some more. Under what conditions will point D be the midpoint of $\overline{BC}$?

Q3 How do BD and CD compare when AB is greater than AC?

To measure a ratio, select two segments. Then, in the Measure menu, choose **Ratio**.

Q4 Your answer to Q3 may give you an idea for a proportion. Measure ratios involving AB, AC, BD, and CD to see if you can create equal ratios (a proportion). Write the proportion below.

7. Manipulate your triangle to make sure your proportion holds for any triangle.

Q5 Write your findings as a conjecture.

Explore More

1. Construct a segment. Use your conjecture to figure out a construction that divides the segment into a given ratio, say 2:3.

Modeling a Pantograph

Name(s): ______________________

A pantograph is a simple mechanical device that uses two pens to copy and enlarge or reduce drawings and maps. Thomas Jefferson made one, hoping he could use it to write more than one letter at a time. In this activity, you'll do a very simple construction that does what a pantograph does. Then, if you're brave and if you have enough time, you'll construct a model that more closely resembles a physical pantograph.

Sketch and Investigate

1. Construct $\overrightarrow{AB}$.
2. Construct point C on $\overrightarrow{AB}$, beyond point B.
3. Select points B and C; then, in the Display menu, turn on **Trace Point.**
4. Drag point B to write your name. Choose **Erase Traces** from the Display menu when you wish to erase all traces from the screen.

Select points A and B; then, in the Measure menu, choose **Distance**. Repeat for AC.

5. Measure AB and AC.

Choose **Calculate** from the Measure menu to open the Calculator. Click once on a measurement to enter it into a calculation.

6. Calculate AB/AC.
7. Draw something with point B. Notice that point C moves on the ray so that the ratio AB/AC stays constant.
8. Move point C to make a different ratio. Experiment with drawing things with point B using different ratios.

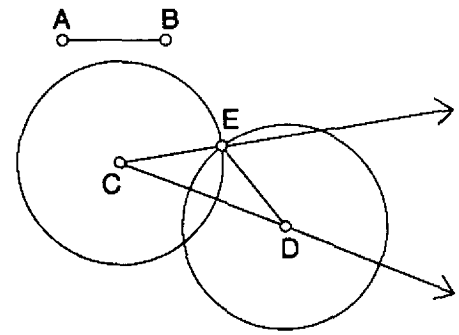
Q1 What does the ratio have to do with the traces of points B and C?

An actual, physical pantograph is constructed of rigid material, such as strips of wood. These pieces don't stretch the way a dynamic Sketchpad ray does. So an actual pantograph depends on linkages that make it flexible.

The following pages describe a construction that models a physical pantograph.

Modeling an Actual Pantograph

9. In a new sketch, construct $\overline{AB}$. (This is not part of the pantograph, but it's a control segment that will make parts of your pantograph both rigid and adjustable.)



Steps 9–15

Hold down the mouse button on the **Segment** tool to show the **Straightedge** palette. Drag right to choose the **Ray** tool.

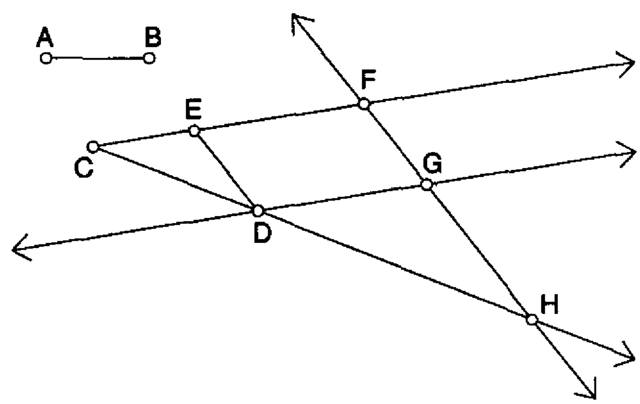
10. Construct $\overrightarrow{CD}$.

Select point C and $\overline{AB}$; then, in the Construct menu, choose **Circle By Center+Radius**.

11. Construct a circle with center point C and radius AB.
12. Construct a circle with center point D and radius AB.
13. Construct point E at one intersection of these circles. (If the circles don't intersect, drag point D until they do.)
14. Construct $\overrightarrow{CE}$.
15. Construct $\overline{DE}$.
16. Hide the circles.
17. Construct $\overline{EF}$ on $\overrightarrow{CE}$.

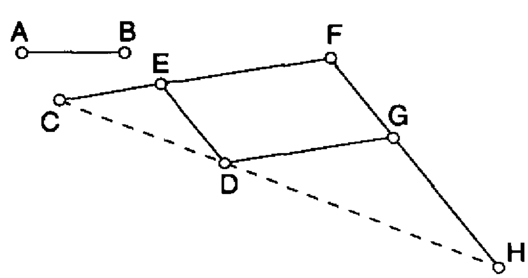
Select point F and $\overline{DE}$; then, in the Construct menu, choose **Parallel Line**.

18. Construct a line through point F parallel to $\overline{DE}$.
19. Construct a line through point D parallel to $\overrightarrow{CE}$.



Steps 16–21

20. Construct point G where these lines intersect.
21. Construct point H at the intersection of $\overleftrightarrow{FG}$ and $\overrightarrow{CD}$.
22. Hide $\overrightarrow{CE}$, $\overrightarrow{CD}$, $\overleftrightarrow{FG}$, and $\overleftrightarrow{DG}$.
23. Construct $\overline{CE}$, $\overline{FG}$, $\overline{DG}$, and $\overline{GH}$. This is something like what a real pantograph looks like.



After you draw $\overline{CD}$, choose **Display | Line Width | Dashed**.

24. Construct $\overline{CD}$ and $\overline{DH}$ and make these segments dashed. These segments wouldn't appear on a real pantograph, but they can help you see how a pantograph works.

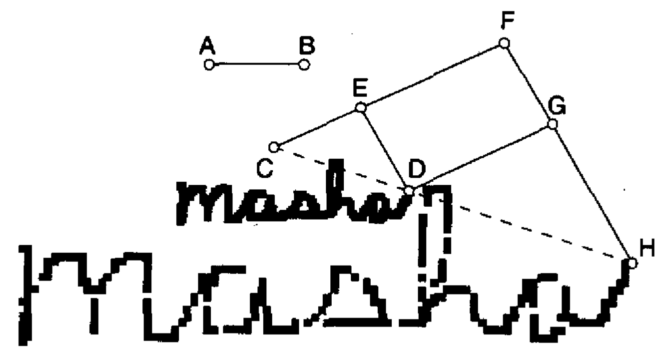
25. Drag point *D* to observe how the pantograph behaves. Note that it falls apart if you drag point *D* too far from point *C*. You can extend its range by lengthening $\overline{AB}$.

26. Turn on **Trace Points** for points *D* and *H*.

This may take several tries. Experiment with different starting places for point *D*. If necessary, make $\overline{AB}$ longer and move point *F* farther from point *E*.

27. Drag point *D* to trace out your name.

28. Move point *F*, then drag point *D* to see how the location of point *F* affects the trace of point *H*.



Q2 How would you locate point *F* so that the trace of point *H* was twice as large as the trace of point *D*? Use similar triangles to explain why.

Explore More

1. Build an actual pantograph out of old rulers, small bolts, and wing nuts.

Proportions with Area

Name(s): ______________________

In this exploration, you'll discover a relationship between the areas of similar figures.

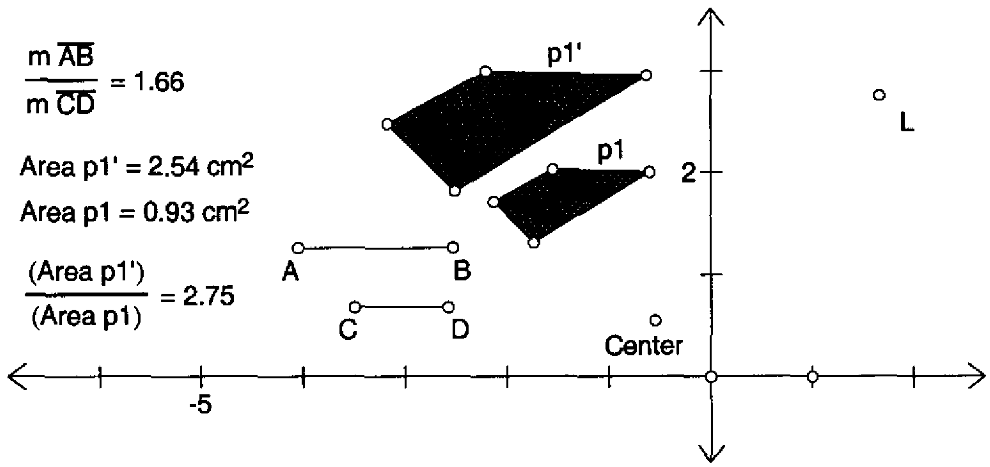
Sketch and Investigate

To construct the interior, select the vertices. Then, in the Construct menu, choose **Polygon Interior**.

1. Construct segments $\overline{AB}$ and $\overline{CD}$, where $\overline{AB}$ is longer than $\overline{CD}$.
2. Construct any polygon and its interior.

Double-click the center point to mark it as a center.

3. Construct a point outside the polygon and mark it as a center.



Select $\overline{AB}$ and $\overline{CD}$; then, in the Transform menu, choose **Mark Segment Ratio**.

Select $\overline{AB}$ and $\overline{CD}$ again, then go to the Measure menu and choose **Ratio**.

4. Mark the ratio of AB to CD.
5. Measure the ratio of AB to CD.

Select the polygon; then, in the Transform menu, choose **Dilate**.

6. Dilate the vertices, sides, and interior of the polygon by the marked ratio. (If $\overline{AB}$ is longer than $\overline{CD}$, the image should be bigger than the original.)
7. Measure the ratio of a side on the dilated polygon to the corresponding side on the original polygon.
8. Measure the ratio of a different pair of corresponding sides.
9. Drag points and observe the ratios you have measured.

Q1 How is the ratio of a pair of corresponding side lengths related to the dilation ratio?

Choose **Calculate** from the Measure menu to open the Calculator. Click once on a measurement to enter it into a calculation.

10. Measure the areas of the polygons.
11. Calculate the ratio of the area of the dilated polygon to the area of the original.

12. Select, in order, the measure of the ratio of the side lengths and the calculation of the ratio of the areas. In the Graph menu, choose **Plot As (x, y)**. You'll get a pair of axes and a point whose coordinates are the two numbers you selected. If you can't see the plotted point, drag the point at (1, 0) closer to the origin to scale the axes.

13. While this point is selected (point L in the diagram on the previous page), choose **Trace Plotted Point** in the Display menu.

14. Drag point B to experiment with different scale factors. The point you plotted will trace out a graph of the side-length ratio versus the ratio of the areas for different similar figures.

Q2 To help you see the relationship between side lengths and areas, complete the table below for some specific side-length ratios, then generalize for any scale factor a/b.

Side-length ratios	2	3	1	1/2	1/10	a/b
Area ratios						

Q3 State your findings as a conjecture.

Q4 Explain how the shape of the graph in your sketch supports your conjecture.

Explore More

1. Move point A close to point C. Merge point B to $\overline{CD}$, then make an action button to animate the point along the segment. Describe what the button does.

2. Drag the center of dilation. Explain why point L (the plotted point) doesn't move.

3. How does the ratio of the volumes of similar solids compare to the ratio of their surface areas and the ratio of their corresponding lengths? Investigate by calculating the volumes and surface areas of two boxes, one twice as long, wide, and tall as the other.