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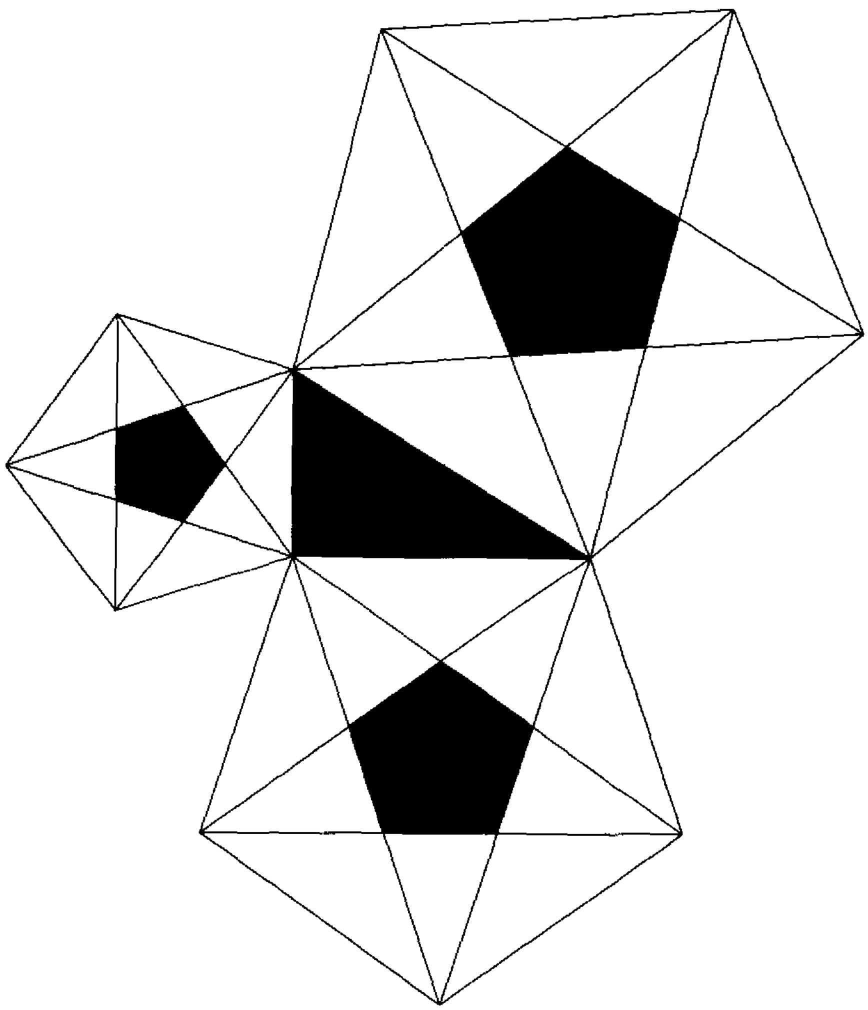
The Pythagorean Theorem



The Pythagorean Theorem

Name(s): ____________________

In this investigation, you'll create a custom tool for constructing a square, then you'll construct squares on the sides of a right triangle. The areas of these squares illustrate perhaps the most famous relationship in mathematics—the Pythagorean theorem.

Sketch and Investigate

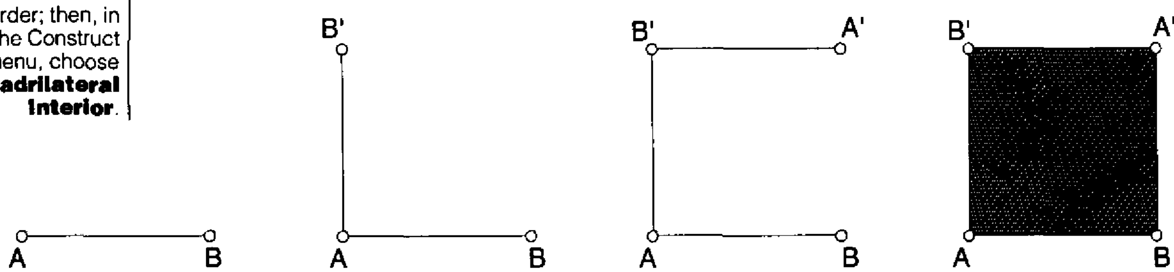
1. Construct $\overline{AB}$.

Double-click point A to mark it as a center. Select point B and $\overline{AB}$; then, in the Transform menu, choose **Rotate**.

2. Mark point A as a center and rotate point B and $\overline{AB}$ by 90°.
3. Mark point B' as a center and rotate point A and $\overline{B'A}$ by 90°.
4. Construct $\overline{A'B}$ to finish the square.

Select the vertices in consecutive order; then, in the Construct menu, choose **Quadrilateral Interior**.

5. Construct the interior.



Step 1 Step 2 Step 3 Steps 4 and 5

6. Drag each vertex of the square to make sure it holds together.

Use the **Text** tool and click on each point.

7. Hide the labels.

Select the entire figure; then, in the Custom Tools menu (the bottom tool in the Toolbox), choose **Create New Tool**.

8. Make a custom tool of this construction.

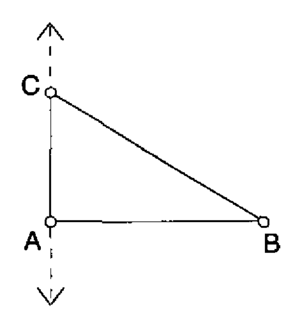
Q1 What properties of a square did you use in this construction?

You might want to save this tool so that you can use it in the future. See the appropriate sections in the help system. (Choose **Toolbox** from the Help menu, then click on the Custom Tools link.)

9. Experiment with using the custom tool to get a feel for the way it works. Note that the direction in which the square is constructed depends on how you use the tool.
10. Open a new sketch.
11. Construct $\overline{AB}$.

Select point A and $\overline{AB}$ and, in the Construct menu, choose **Perpendicular Line**.

12. Construct a line through point A perpendicular to $\overline{AB}$.
13. Construct $\overline{BC}$, where point C is a point on the perpendicular line.



14. Hide the perpendicular line and construct $\overline{AC}$.

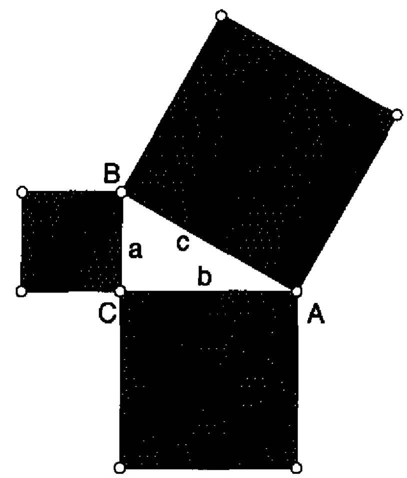
15. Drag each vertex to confirm that your triangle stays a right triangle.

Q2 What property of a right triangle did you use in your construction?

To change a label, double-click the label with the finger of the **Text** tool. (The reason for changing these labels is so that your figure will match the way the theorem is usually stated. This may make it easier to remember the theorem.)

16. Change labels so that the right-angle vertex is labeled C and the other two vertices are labeled *A* and *B*.

17. Show the labels of the sides. Change them to *a, b,* and *c* so that side *a* is opposite $\angle A$, side *b* is opposite $\angle B$, and side *c* is opposite $\angle C$.



Be sure to attach each square to a pair of the triangle's vertices. If your square goes the wrong way (overlaps the interior of your triangle) or is not attached properly, undo and try attaching the square to the triangle's vertices in the opposite order.

18. Use your square tool to construct squares on the sides of your triangle.

19. Drag the vertices of the triangle to make sure the squares are properly attached.

20. Measure the areas of the three squares.

21. Measure the lengths of sides *a, b,* and *c*.

22. Drag each vertex of the triangle and observe the measures.

Choose **Calculate** from the Measure menu to open the Calculator. Click once on a measurement to enter it into a calculation.

Q3 Describe any relationship you see among the three areas. Use the Calculator to create an expression that confirms your observations.

Q4 Based on your observations about the areas of the squares, write an equation that relates *a, b,* and *c* in any right triangle. (*Hint:* What's the area of the square with side length *a*? What are the areas of the squares with side lengths *b* and *c*? How are these areas related?)

Explore More

1. Do a similar investigation using other figures besides squares. Does your conjecture about the areas still hold?

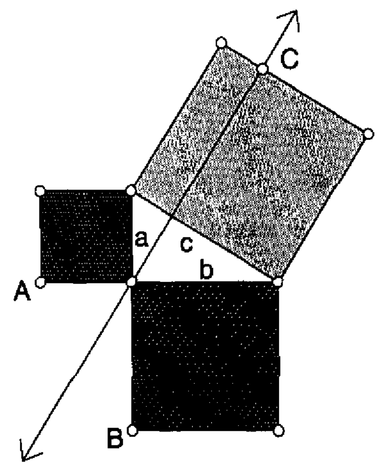
2. Investigate the converse of the Pythagorean theorem: Construct a nonright triangle and squares on its sides. Measure the areas of the squares and sum two of them. Drag until the sum is equal to the third area. What kind of triangle do you have?

Visual Demonstration of the Pythagorean Theorem

Name(s): ____________________

In this activity, you'll do a visual demonstration of the Pythagorean theorem based on Euclid's proof. By *shearing* the squares on the sides of a right triangle, you'll create congruent shapes without changing the areas of your original squares.

Sketch and Investigate



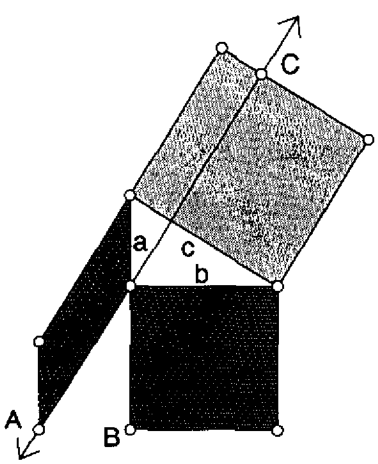
1. Open the sketch **Shear Pythagoras.gsp.** You'll see a right triangle with squares on its sides.

Click on an interior to select it. Then, in the Measure menu, choose **Area**.

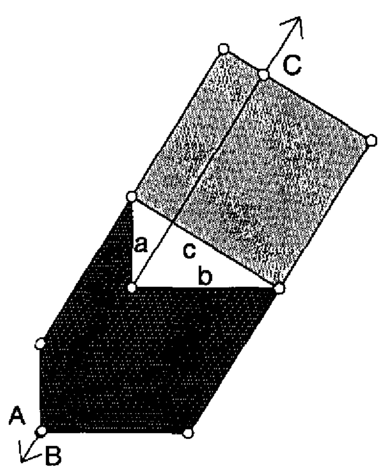
2. Measure the areas of the squares.
3. Drag point A onto the line that's perpendicular to the hypotenuse. Note that as the square becomes a parallelogram its area doesn't change.
4. Drag point B onto the line. It should overlap point A so that the two parallelograms form a single irregular shape.

To confirm that this shape is congruent, you can copy and paste it. Drag the pasted copy onto the shape on the legs to see that it fits perfectly.

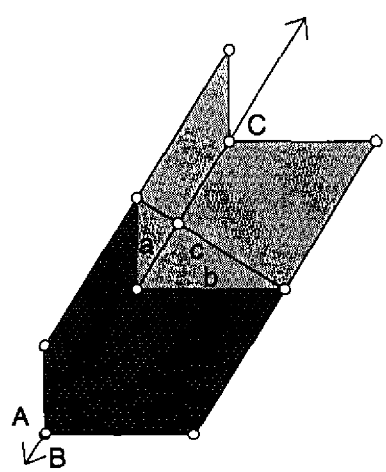
5. Drag point C so that the large square deforms to fill in the triangle. The area of this shape doesn't change either. It should appear congruent to the shape you made with the two smaller parallelograms.



Step 3



Step 4



Step 5

To confirm that this works for any right triangle, change the shape of the triangle and try the experiment again.

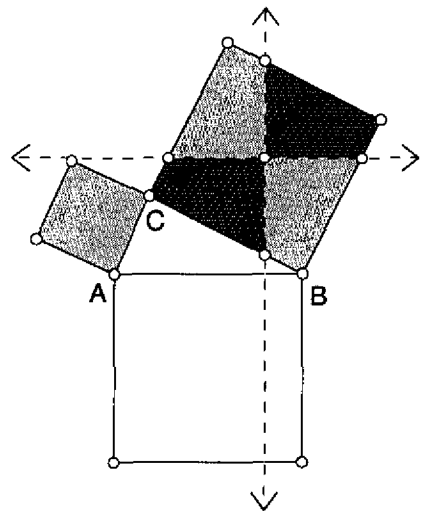
Q1 How do these congruent shapes demonstrate the Pythagorean theorem? (*Hint:* If the shapes are congruent, what do you know about their areas?)

Dissection Demonstration of the Pythagorean Theorem

Name(s): ____________________

Many demonstrations of the Pythagorean theorem involve cutting up the squares on the legs of a right triangle and rearranging them to fit into the square on the hypotenuse. These demonstrations are called *dissections*. Some people might consider a dissection demonstration a proof. Others would require an explanation of why the dissection works.

Sketch and Investigate



1. Construct a right triangle ABC. Drag vertices to make sure your triangle is properly constructed.

You can use the tool **4/Square (By Edge)** from the sketch **Polygons.gsp**.

2. Use a custom tool to construct squares on the sides. Delete the interiors if necessary.

One way to construct the center is to construct a diagonal and its midpoint. Hide the diagonal.

3. Construct the center of the square on the longer leg.
4. Construct a line through this center, parallel to the hypotenuse.
5. Construct another line through the center, this time perpendicular to the hypotenuse.
6. Construct the four points where these lines intersect the sides of the square. Hide the lines.

Select the vertices of the polygon in consecutive order; then, in the Construct menu, choose **Quadrilateral Interior**.

7. Construct four interiors in this square as shown, using the center as one vertex.
8. Construct the interior of the square on the small leg.

You now have five interiors: four in the large square plus the one small square. Can these five pieces be rearranged to fit in the square on the hypotenuse? Follow steps 9–11 to find out.

9. Select the five interiors. In the Edit menu, choose **Cut**.
10. In the Edit menu, choose **Paste**. The pasted interiors are now free, and you can move them around.

Before you drag, click in a blank area to deselect everything. Then you can drag one piece at a time.

11. Drag each piece into the square on the hypotenuse and arrange them so they fill this square without gaps or overlapping.

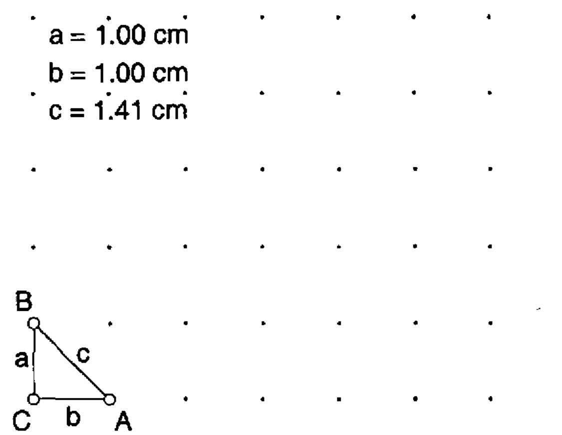
Q1 What does this demonstrate about the sides of the right triangle?

Pythagorean Triples

Name(s): ______________________

The Pythagorean theorem states that if a right triangle has side lengths a and b and hypotenuse length c, then $a^2 + b^2 = c^2$. A set of three whole numbers that satisfy the Pythagorean theorem is called a Pythagorean triple. In this activity, you'll find as many right triangles as you can whose side lengths are whole numbers.

Sketch and Investigate



1. Make your sketch window as large as you can.

In the Edit menu, choose **Preferences** and go to the Units panel.

2. In Preferences, set the Distance Units to **cm** and the Distance Precision to **hundredths**.

In the Graph menu, choose **Show Grid**, then choose **Snap Points**. Select the grid (by clicking on a grid intersection) and choose **Display | Line Width | Dotted**.

3. Show the grid and turn on point snapping.
4. Hide the axes and the two control points.
5. In the lower left corner of your sketch, draw a right triangle *ABC* with vertices on the grid.

Using the **Text** tool, click on a segment to show its label. Double-click a label to change it.

6. Show the segment labels and change them to *a*, *b*, and *c*, as shown.
7. Measure the three side lengths.
8. Make the two leg lengths 1 cm each, as shown.

Q1 In this case, you can see that the hypotenuse length is not a whole number. Use the Pythagorean theorem to find the exact hypotenuse length (in radical form) when the side lengths are 1 cm. Show your work.

9. Drag point *A* one unit to the right.

Q2 When the leg lengths are 1 cm and 2 cm, is the hypotenuse length a whole number?

10. Drag point *A* one unit to the right again and look to see if the hypotenuse length is a whole number.

You may or may not be able to fill in the whole chart, depending on the thoroughness of your search and the size of your screen. If your screen is very large, you may even need to add rows to the chart.

Q3 Continue a systematic search for Pythagorean triples, dragging point A one unit at a time to the right to increase b and dragging point B one unit up to increase a. Any time c is a whole number, record the Pythagorean triple in the chart at right.

a	b	c

Refer to your chart and experiment with the sketch to answer the following questions:

Q4 Which sets of triples are side lengths of congruent triangles?

Q5 Which sets of triples are side lengths of similar triangles (triangles with the same shape)?

Q6 Do you think there is a limit to the number of Pythagorean triples possible? Explain.

Q7 In the space below, use the Pythagorean theorem to verify at least three of your sets of triples.

Explore More

1. Euclid's *Elements* demonstrates that Pythagorean triples can be generated by the formulas $m^2 - n^2$, $2mn$, $m^2 + n^2$, where m and n are positive integers and m is greater than n. What triple is generated by $m = 2$ and $n = 1$? Increase m and n and generate some other triples. Can you generate all the triples you recorded in your chart? Can you generate some triples that aren't on your chart? Draw some triangles with these side lengths on the grid to confirm that they're right triangles.

The Isosceles Right Triangle

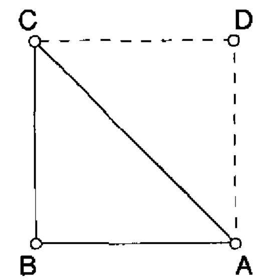
Name(s): ____________________

In this activity, you'll discover a relationship among the side lengths of an isosceles right triangle. This relationship will give you a shortcut for finding side lengths quickly. You'll start by constructing a square. Dividing this square in half along a diagonal gives you the isosceles right triangle.

Sketch and Investigate

You can use the tool **4/Square (By Edge)** from the sketch **Polygons.gsp**.

1. Use a custom tool to construct a square *ABCD* by edge endpoints *A* and *B*.



2. Construct diagonal *CA*.

Hide and **Line Width** are in the Display menu.

3. Hide the square's interior, if it has one.
4. Change the line widths of $\overline{CD}$ and $\overline{DA}$ to dashed.

Q1 Explain why ΔABC is an isosceles right triangle.

Q2 Without measuring, state the measures of the acute angles in ΔABC.

5. Measure the three side lengths.

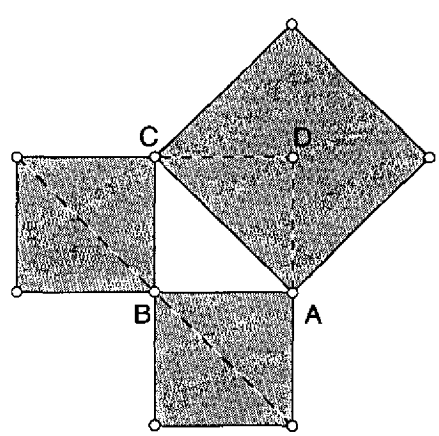
Select the hypotenuse and one of the legs. Then, in the Measure menu, choose **Ratio**.

6. Measure the ratio of the hypotenuse length to one of the leg lengths.
7. Drag point *A* or point *B* and observe this ratio.

Q3 What do you notice about this ratio?

In steps 8 and 9, you'll investigate what this ratio represents geometrically.

8. Use the square tool to construct squares on the sides of right triangle *ABC*. Drag to make sure the squares are properly attached.
9. Construct one diagonal in each of the smaller squares, as shown at right.



Q4 The diagonals you drew in the smaller squares may help you see a relationship between the smaller squares and the square on the hypotenuse. If each of the smaller squares has area x^2, what is the area of the large square? Confirm your conjecture by measuring the areas. Drag the triangle to confirm that this relationship always holds.

Q5 In an isosceles right triangle, if the legs have length x, what is the length of the hypotenuse?

Q6 Use the Pythagorean theorem to confirm your answer to Q5.

Explore More

1. The society of the Pythagoreans discovered that the square root of 2 is *irrational*. Do some research and report on this discovery.

The 30°-60° Right Triangle

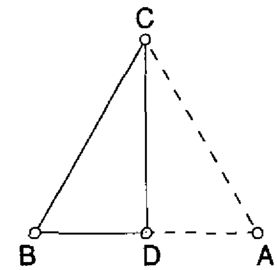
Name(s): ____________________

The 30°-60° right triangle—formed by taking half of an equilateral triangle—has special relationships among its side lengths. These relationships make it easy to find all the side lengths if you know just one. In this activity, you'll discover these relationships and why they hold.

Sketch and Investigate

You can use the tool **3/Triangle (By Edge)** from the sketch **Polygons.gsp**. If you need to relabel the triangle, use the **Text** tool. Click once on a point to show its label. Double-click a label to change it.

1. In a new sketch, construct an equilateral triangle *ABC*. Use a custom tool or construct it from scratch. Drag vertices to confirm that the construction is correct.



2. Hide the triangle's interior if necessary.
3. Construct the midpoint *D* of $\overline{AB}$.
4. Construct median *CD*.

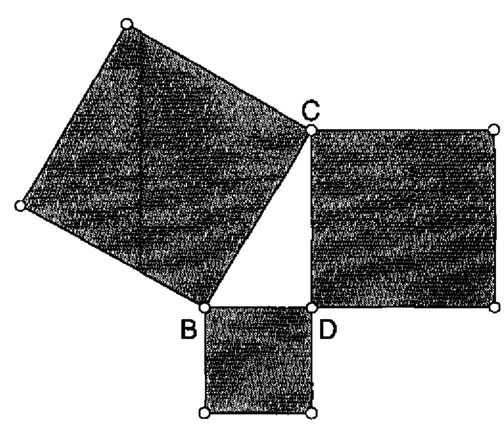
Select the segments; then, in the Display menu, choose **Line Width | Dashed**.

5. Change the line widths of $\overline{AB}$ and $\overline{AC}$ to dashed.
6. Construct $\overline{BD}$ and make its line width thin.

Q1 Without measuring, state the measure of each angle in ΔCDB. For each angle, explain how you know it has that measure.

Q2 In a 30°-60° right triangle, how does the length of the hypotenuse compare to the length of the short leg? Answer without measuring.

7. Hide point *A*, $\overline{AB}$, and $\overline{AC}$.
8. Use a custom tool to construct squares on the three sides of the triangle as shown at right. Drag to make sure the squares are properly attached.



9. Measure the areas of the three squares.

Choose **Calculate** from the Measure menu to open the Calculator. Click once on a measurement to enter it into a calculation.

10. Calculate the ratio of the largest area to the smallest area
11. Drag point *B* and observe this ratio.

Q3 What is this area ratio? Explain why this ratio is what it is. In your explanation, use what you know about the side lengths.

Q4 Now you'll use your answer to Q3 about the square on the hypotenuse and the square on the short leg to help you find the area of the square on the long leg. Answer the following questions:

a. Suppose the smallest square has area x^2. What would be the area of the square on the hypotenuse? __________

b. Use the Pythagorean theorem to find the area of the square on the long leg. Show your work.

c. State the ratio of the area of the square on the long leg to the area of the square on the short leg. Calculate this ratio in your sketch and drag point B to confirm that this ratio applies to all 30°-60° right triangles. __________

Q5 Suppose the short leg had length x.

a. What would be the length of the hypotenuse? __________

b. What would be the length of the long leg? __________

c. What would be the ratio of the length of the long leg to the length of the short leg? (State your answer in radical form.) __________

Select the hypotenuse and the short leg. Then, in the Measure menu, choose **Ratio**. Measure the other ratio in the same way.

12. To confirm your answers to Q5, measure the ratio of the hypotenuse length to the short leg length. Also measure the ratio of the long leg length to the short leg length. Drag point B to confirm that these ratios apply to all 30°-60° right triangles.

Q6 The second ratio you measured in step 12 should be the decimal approximation of the ratio you wrote in Q5c. Write this decimal approximation. __________

Explore More

1. To test how well you can apply your discoveries, make a Hide/Show action button for each side length measurement. Show one side length and hide the other two. Then calculate the two hidden lengths. Show the hidden lengths to check your calculations. Try this several times, changing the triangle each time and showing different side lengths. Repeat until you think you can calculate the missing side lengths correctly every time.

The Square Root Spiral

Name(s): ____________________

Irrational numbers such as $\sqrt{2}$ and $\sqrt{3}$ correspond to points on a ruler, but you can't find those points precisely by dividing your ruler into fractional parts. However, you can construct square roots with compass and straightedge (or with Sketchpad). In this activity, you'll construct a square root spiral and use it to create a chart of approximate square roots.

Sketch and Investigate

The first part of the spiral is an isosceles right triangle.

Choose **Preferences** from the Edit menu and go to the Units panel.

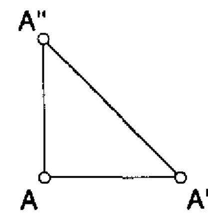
1. In Preferences, set the Distance Units to **inches** and set Distance Precision to **thousandths**.

With point *A* selected, choose **Translate** in the Transform menu.

2. Construct point *A* and translate it at an angle of 0° by 1 inch to create point A'.

Double-click point *A* to mark it as a center. Select point A'; then, in the Transform menu, choose **Rotate**.

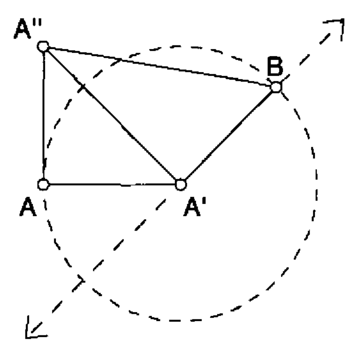
3. Mark point *A* as a center and rotate point A' by 90°.
4. Connect these points to make isosceles right triangle $AA'A''$.



Q1 Use the Pythagorean theorem to find the exact value of $A'A''$. (Use radical form, not a decimal approximation, and show your work.)

In steps 5–9, you'll construct $\sqrt{3}$.

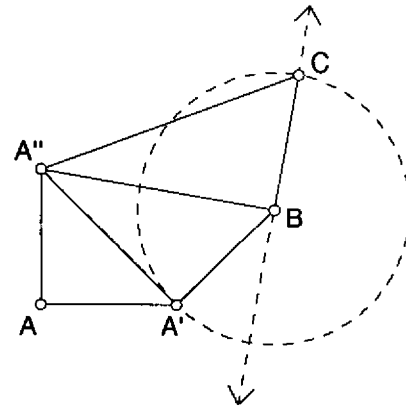
5. Construct a line through point A', perpendicular to $\overline{A'A''}$.
6. Construct circle $A'A$.
7. Construct point *B*, the intersection of the circle with the line, as shown.
8. Hide the circle and line.
9. Construct $\overline{A'B}$ and $\overline{A''B}$.



Q2 Explain why $A''B$ is equal to $\sqrt{3}$ inches.

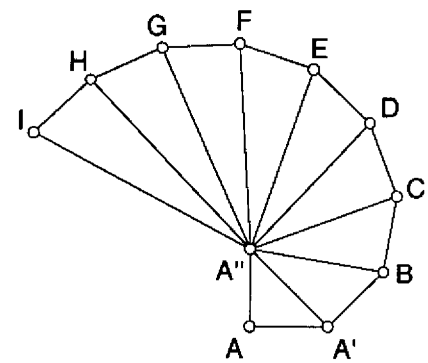
You can continue in this way to construct a spiral that gives you the square roots of as many consecutive positive integers as you like. A custom tool makes the construction process quicker. Follow steps 10–17 to continue your spiral.

10. Construct a line through point B, perpendicular to $\overline{BA''}$.
11. Construct circle BA'.
12. Construct point C, the intersection of the circle with the line, as shown.
13. Hide the circle and line.
14. Construct segments $A''C$ and CB.



Selection order is very important to make the custom tool work.

15. Now create a custom tool. Select, in order, point A', point B, point A'' (these are the tool's *givens*), point C, $\overline{A''C}$, and $\overline{BC}$ (these are the tool's *results*). Choose **Create New Tool** from the Custom Tools menu (the bottom tool in the Toolbox) and name the tool **Next Triangle**.
16. Click on the **Custom** tools icon to select the tool you just created. Click, in order, on point B, point C, and point A'' to create the next right triangle.
17. Use your custom tool to create at least five more right triangles.



Save this sketch. You can use its custom tool again in Explore More 1, below.

Q3 What are the lengths of all the segments around the outside of the spiral? Write these lengths on the figure at right.

Q4 Using radical form, write the lengths of the spiral arms on the figure above right.

Q5 Measure all the spiral arms, then complete the table below with approximations to the nearest thousandth for square roots of whole numbers from 2 to 10.

$\sqrt{1} = 1$	$\sqrt{2} \approx$	$\sqrt{3} \approx$	$\sqrt{4} =$	$\sqrt{5} \approx$
$\sqrt{6} \approx$	$\sqrt{7} \approx$	$\sqrt{8} \approx$	$\sqrt{9} =$	$\sqrt{10} \approx$

Explore More

1. Draw an arbitrary triangle. Use the **Next Triangle** tool on its three vertices. Now use the tool, in the same order, on the vertices of the newly created triangle. Do this at least eight more times. You should get a spiral (though not a square root spiral) that you can open and close dynamically by dragging the vertices of your original triangle.