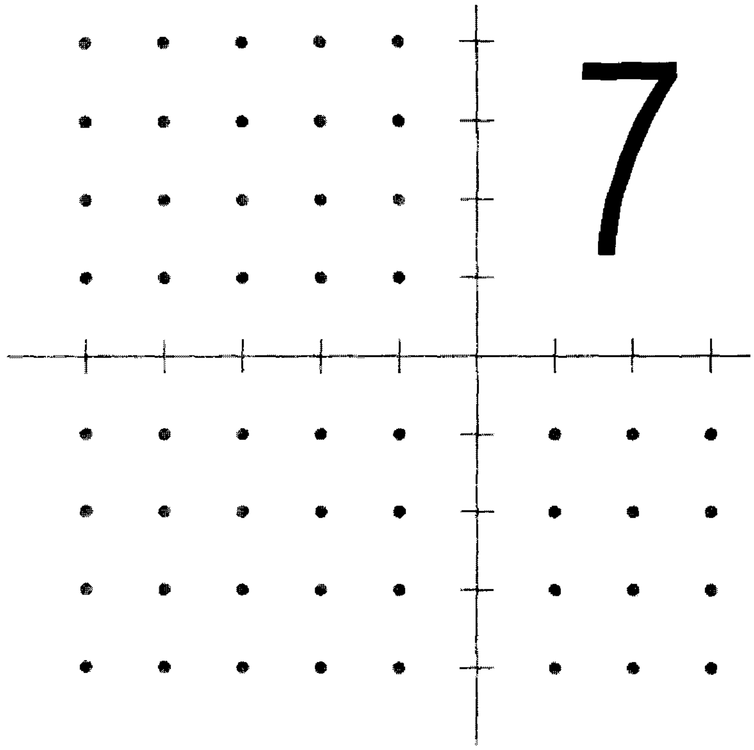
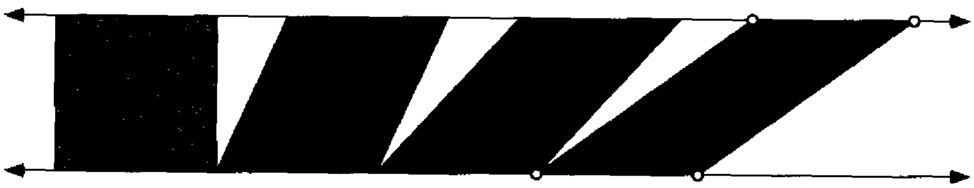


7

Area



Areas of Parallelograms and Triangles

Name(s): ______________________

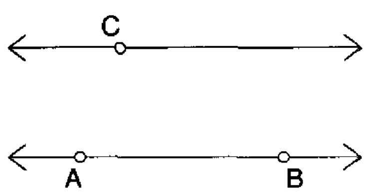
You'll discover a relationship between the areas of parallelograms and triangles by investigating a process called *shearing*. This will give you a formula for area that you can generalize for all parallelograms.

Sketch and Investigate

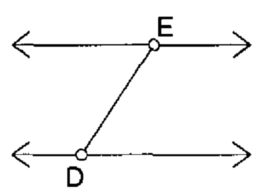
1. Construct a horizontal line AB.
2. Construct point C above $\overleftrightarrow{AB}$.
3. Construct a line parallel to $\overleftrightarrow{AB}$ through point C.

Select point C and $\overleftrightarrow{AB}$; then, in the Construct menu, choose **Parallel Line**.

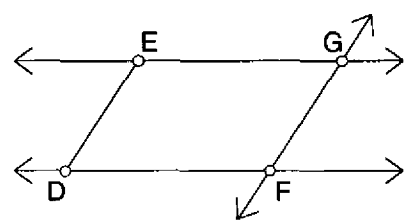
4. Hide points A, B, and C.
5. Construct $\overline{DE}$ from the bottom line to the top line.
6. Construct point F on the bottom line.
7. Construct a line through point F parallel to $\overline{DE}$.
8. Construct point G where this line intersects the top line.



Steps 1–3



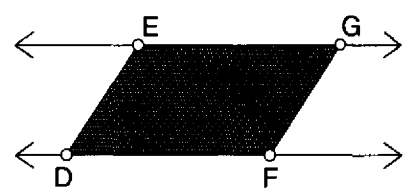
Steps 4 and 5



Steps 5–7

Select the vertices in consecutive order; then, in the Construct menu, choose **Quadrilateral Interior**.

9. Construct interior $DEGF$.
10. Hide $\overleftrightarrow{FG}$.
11. Construct $\overline{FG}$.

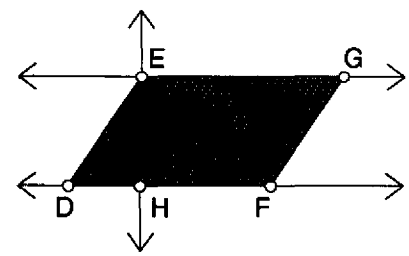


Select the interior by clicking on it; then, in the Measure menu, choose **Area**.

12. Measure the area of parallelogram $DEGF$.
13. Observe the area measurement as you drag in each of these ways:
    - Drag point E to shear the parallelogram.
    - Drag point D or point F to change the base of the parallelogram.
    - Drag either $\overleftrightarrow{EG}$ or $\overleftrightarrow{DF}$ up or down to change the height.

Q1 Which of these actions change the area and which don't? Explain why you think this is so.

The height of the parallelogram is the distance between the two parallel lines. To construct a segment whose length is this height, follow steps 14 and 15.



14. Construct a line through point E perpendicular to $\overleftrightarrow{DF}$.
15. Construct $\overline{EH}$, where H is the point of intersection of $\overleftrightarrow{DF}$ and the perpendicular line.
16. Hide the perpendicular line.
17. Measure EH.
18. Measure DF, the base of the parallelogram.

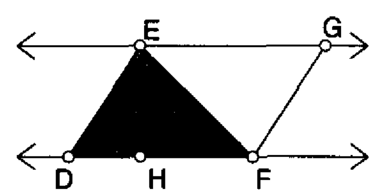
Select the two endpoints; then, in the Measure menu, choose **Distance**.

Q2 Use these measurements to calculate an expression equal to the area of the parallelogram. Write your expression below.

Choose **Calculate** from the Measure menu to open the Calculator. Click once on a measurement to enter it into a calculation.

Q3 Write a formula for the area of the parallelogram using A for area, b for base, and h for height.

Next, you'll investigate how the area of the parallelogram is related to the area of a triangle.



19. Hide the interior of the parallelogram.
20. Construct diagonal EF.
21. Construct polygon interior DEF.
22. Measure the area of triangle DEF.
23. Drag point E and observe the area measurements.

Q4 How is the triangle's area related to the parallelogram's area? Write a formula for the area of the triangle using A for area, b for base, and h for height.

Explore More

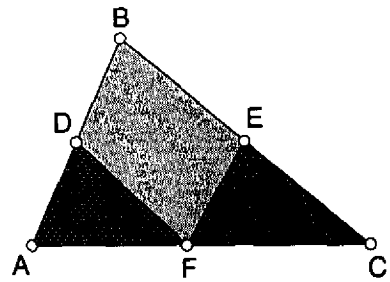
1. Make an action button to animate point E along its line. Explain what this animation demonstrates about shearing.

A Triangle Area Problem

Name(s): ______________________

In this investigation, you'll divide a triangle into regions and explore the relationship among these areas.

Sketch and Investigate



1. Construct $\triangle ABC$.
2. Construct points D and E, the midpoints of $\overline{AB}$ and $\overline{BC}$.
3. Construct $\overline{DF}$ and $\overline{EF}$, where point F is any point on $\overline{AC}$.
4. Construct the polygon interiors of triangles ADF and FEC and quadrilateral $BDFE$.

Select the vertices of the polygon in consecutive order; then, in the Construct menu, choose **Triangle Interior** or **Quadrilateral Interior**.

Q1 Move point F back and forth along $\overline{AC}$. Without measuring, guess how the areas of the triangles are related to the area of the quadrilateral. (*Hint:* Look at special cases, such as when F is at the midpoint, point A, and point C.)

5. Measure the areas of the triangles and the quadrilateral to check your guess.

Q2 Write a conjecture about how the areas of the triangles are related to the area of the quadrilateral.

Q3 Explain why you think your conjecture is always true.

Explore More

1. Make an action button to animate point F along $\overline{AC}$.
2. Here's another challenging area problem: Construct a square. Find two segments from one of the vertices that divide the square into three equal areas. Where should the other endpoints of these segments be located?

Triangle Area/Perimeter

Name(s): ____________________

Is it possible to construct two noncongruent triangles that have both equal areas and equal perimeters? Use Sketchpad to investigate this question. Use the space below to describe your findings, or print a sketch with comments that describe your findings.

To measure area and perimeter, you must first construct an interior. To construct an interior, select the vertices in consecutive order. Then, in the Construct menu, choose **Triangle Interior**.

Q1 If possible, draw and label two noncongruent triangles that have equal areas and perimeters in the space below. (Or print your sketch and attach it to this paper.) If you find it's not possible, illustrate why.

Q2 In the space below, write a description of what you did in this investigation. If you created two noncongruent triangles with equal areas and perimeters, describe how you did it.

A Square Within a Square

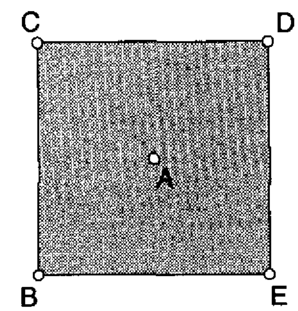
Name(s): ____________________

In this investigation, you'll construct a square and four segments within it that intersect to form a second square. Then you'll discover an interesting area relationship between these squares.

Sketch and Investigate

You can use the tool **4/Square (Inscribed)** from the sketch **Polygons.gsp**.

1. Construct a square with its center and its interior. Use a custom tool or do the construction from scratch.

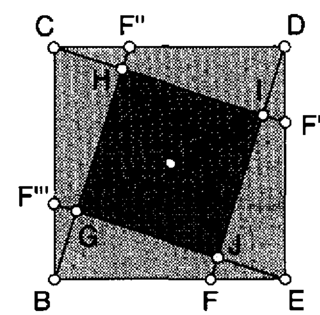


Using the **Text** tool, click once on a point to display its label. Double-click the label to change it.

2. Change the labels, if necessary, to match the figure at right.
3. Construct point F on $\overline{BE}$, closer to point E.

Double-click point A to mark it as a center. Select point F; then, in the Transform menu, choose **Rotate**.

4. Mark point A as a center and rotate point F by 90° to construct point F'.
5. Rotate point F' by 90° to construct point F'', then rotate point F'' by 90° to construct F'''.
6. Construct $\overline{BF''}$, $\overline{CF'}$, $\overline{DF}$, and $\overline{EF'''}$ as shown in the figure at right.



Select the vertices in consecutive order; then, in the Construct menu, choose **Quadrilateral Interior**.

7. Construct points of intersection G, H, I, and J and interior $GHIJ$. Give it a darker color than the outside square interior.
8. Measure the areas of the outer and inner squares.

Choose **Calculate** from the Measure menu to open the Calculator. Click once on a measurement to enter it into a calculation.

9. Calculate the ratio of the larger area to the smaller area.
10. Drag point F and observe this ratio. Make a guess as to what the ratio would be if point F were at the midpoint.

It's difficult to drag point F exactly to the midpoint. For that reason, you'll construct the midpoint and other interesting points and make Movement buttons to move point F to precise locations.

11. Construct the midpoint of $\overline{BD}$ and change its label to 1/2.

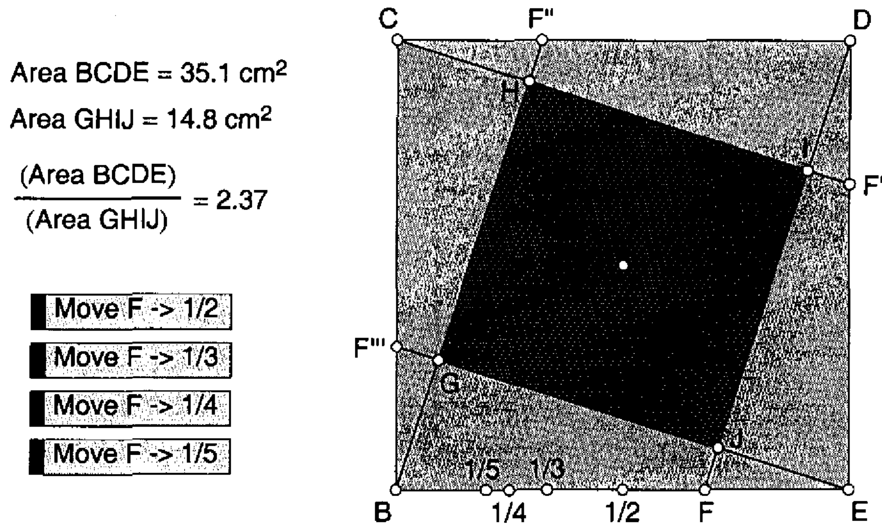
Select, in order, point F and point 1/2. Then choose **Edit | Action Buttons | Movement**.

12. Make a Movement button to move point F to point 1/2.
13. Press the *Move F —> 1/2* button.

Q1 What is the area ratio? Is it what you expected?

After you mark the center, select point *E* and, in the Transform menu, choose **Dilate**. Enter 1 for the numerator of the scale factor and 3 for the denominator.

14. Mark point *B* as a center and dilate point *E* by a scale factor of 1/3. Change the label of the dilated point to 1/3.
15. Construct 1/4 and 1/5 points in the same way.
16. Make buttons to move point *F* to point 1/3, to point 1/4, and to point 1/5.



17. Use your Movement buttons to investigate the other ratios.

Q2 Write the area ratios for side-divisions of 1/3, 1/4, and 1/5.

Q3 Look for a pattern in the area ratios and predict the area ratio for a side-division ratio of 1/6.

Explore More

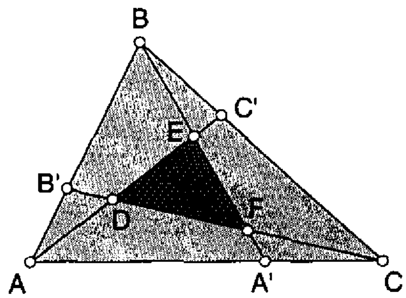
1. Write an expression that gives the area ratio when the side is divided into a ratio of $1/n$.

A Triangle Within a Triangle

Name(s): ____________________

If you connect each vertex of a triangle with a trisection point on the opposite side, you'll form a second triangle within the first. In this activity, you'll investigate a relationship between the two triangles.

Sketch and Investigate



1. Construct ΔABC.

Double-click point *A* to mark it as a center. Select point *B*; then, in the Transform menu, choose **Dilate**. Enter 1 for the numerator of the scale factor and 3 for the denominator.

2. Mark point *A* as a center and dilate point *B* by a scale factor of 1/3.
3. Mark point *B* as a center and dilate point *C* by a scale factor of 1/3.
4. Mark point *C* as a center and dilate point *A* by a scale factor of 1/3.
5. Construct segments from each vertex to the 1/3 point on the opposite side.
6. Construct points of intersection *D*, *E*, and *F*. These points are the vertices of an inner triangle formed by the segments you constructed in step 5.

Select the triangle's vertices; then, in the Construct menu, choose **Triangle Interior**.

7. Construct the interiors of the outer and inner triangles. Make the outer triangle a lighter shade.
8. Drag a vertex of the outer triangle and observe the inner triangle. Before you measure, try to guess the relationship between the larger and smaller areas.

Click on an interior to select it; then, in the Measure menu, choose **Area**.

9. Measure the areas of the outer and inner triangles.

Choose **Calculate** from the Measure menu to open the Calculator. Click once on a measurement to enter it into a calculation.

10. Calculate the ratio of the larger area to the smaller. Surprised?
11. Drag a vertex of the outer triangle and observe the area ratio.

Q1 Write a conjecture about the area ratio.

Explore More

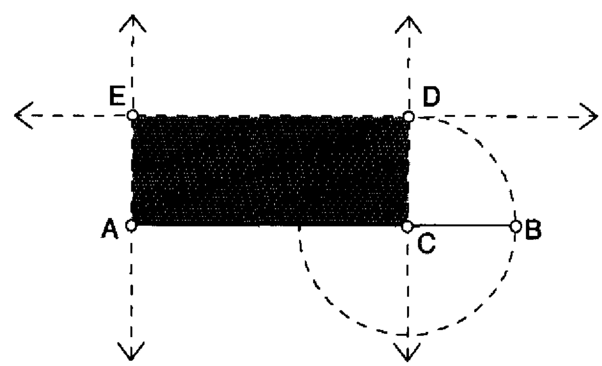
1. Investigate whether or not the large and small triangles are similar.
2. Try this investigation again, subdividing the sides of the triangle into fourths.
3. Try this investigation on a quadrilateral.

A Rectangle with Maximum Area

Name(s): ____________________

Suppose you had a certain amount of fence and you wanted to use it to enclose the biggest possible rectangular field. What rectangle shape would you choose? In other words, what type of rectangle has the most area for a given perimeter? You'll discover the answer in this investigation. Or, if you have a hunch already, this investigation will help confirm your hunch and give you more insight into it.

Sketch and Investigate



1. Construct $\overline{AB}$.

Select $\overline{AB}$, point *A*, and point *C*. Then, in the Construct menu, choose **Perpendicular Line**.

2. Construct $\overline{AC}$ on $\overline{AB}$.
3. Construct lines perpendicular to $\overline{AB}$ through points *A* and *C*.

Be sure to release the mouse—or click the second time—with the pointer over point *B*.

4. Construct circle *CB*.
5. Construct point *D* where this circle intersects the perpendicular line.
6. Construct a line through point *D*, parallel to $\overline{AB}$.
7. Construct point *E*, the fourth vertex of rectangle *ACDE*.

Select the vertices of the rectangle in consecutive order. Then, in the Construct menu, choose **Quadrilateral Interior**.

8. Construct interior *ACDE*.
9. Measure the area and perimeter of this polygon.
10. Drag point *C* back and forth and observe how this affects the area and perimeter of the rectangle.

Select point *A* and point *C*. Then, in the Measure menu, choose **Distance**. Repeat to measure *AE*.

11. Measure *AC* and *AE*.

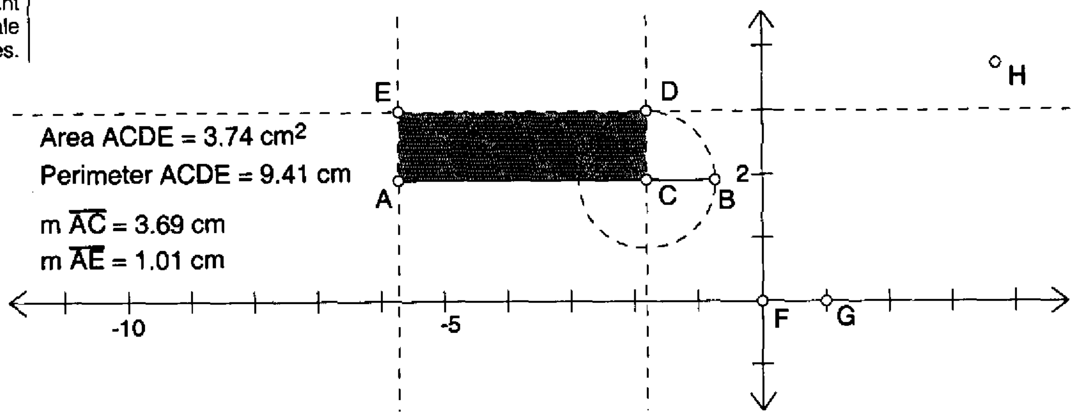
Q1 Without measuring, state how *AB* is related to the perimeter of the rectangle. Explain why this rectangle has a fixed perimeter.

Q2 As you drag point *C*, observe what rectangular shape gives the greatest area. What shape do you think that is?

In steps 12–14, you'll explore this relationship graphically.

Select, in order, measurements AC and Area $ACDE$. Then choose **Plot As (x, y)** from the Graph menu. If you can't see the plotted point, drag the unit point at (1, 0) to scale the axes.

12. Plot the measurements for the length of $\overline{AC}$ and the area of $ACDE$ as (x, y). You should get axes and a plotted point H, as shown below.
13. Drag point C to see the plotted point move to correspond to different side lengths and areas.



Select point H and point C; then, in the Construct menu, choose **Locus**.

14. To see a graph of all possible areas for this rectangle, construct the locus of plotted point H as defined by point C. It should now be easy to position point C so that point H is at a maximum value for the area of the rectangle.

You may wish to select point H and measure its coordinates.

Q3 Explain what the coordinates of the high point on the graph are and how they are related to the side lengths and area of the rectangle.

15. Drag point C so that point H moves back and forth between the two low points on the graph.

Q4 Explain what the coordinates of the two low points on the graph are and how they are related to the side lengths and area of the rectangle.

Explore More

1. Investigate area/perimeter relationships in other polygons. Make a conjecture about what kinds of polygons yield the greatest area for a given perimeter.
2. What's the equation for the graph you made? Let AC be x and let AB be $(1/2)P$, where P stands for the perimeter (a constant). Write an equation for area, A, in terms of x and P. What value for x (in terms of P) gives a maximum value for A?

The Area of a Trapezoid

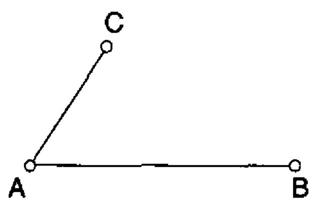
Name(s): ______________________

A trapezoid is a quadrilateral with exactly two parallel sides. In this investigation, you'll construct a trapezoid, then transform it into a shape whose area formula you should be familiar with. From that formula, you'll derive a formula for the area of a trapezoid.

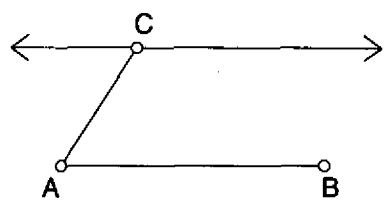
Sketch and Investigate

1. Construct $\overline{AB}$.
2. Construct $\overline{AC}$.
3. Construct a line through point C parallel to $\overline{AB}$.
4. Construct $\overline{DB}$, where point D is a point on the line.

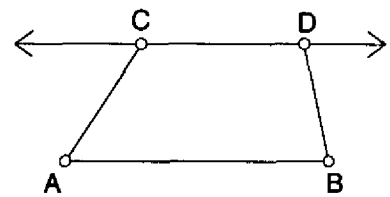
Select point C and $\overline{AB}$; then, in the Construct menu, choose **Parallel Line**.



Steps 1 and 2



Step 3



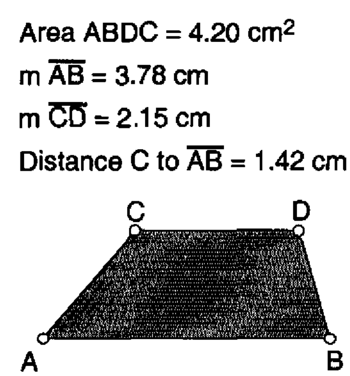
Step 4

5. Hide the line.
6. Construct $\overline{CD}$.
7. Construct the interior of trapezoid $ABDC$.
8. Measure the area of $ABDC$.
9. Measure the lengths of the bases of the trapezoid, $\overline{AB}$ and $\overline{CD}$.
10. Measure the distance from point C to $\overline{AB}$. This is the height of the trapezoid.

Select the line; then, in the Display menu, choose **Hide**.

Select the vertices in consecutive order; then, in the Construct menu, choose **Quadrilateral Interior**.

Select point C and $\overline{AB}$; then, in the Measure menu, choose **Distance**.

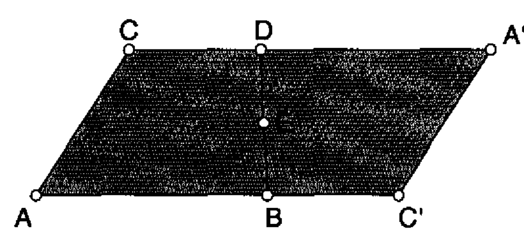


11. Drag different parts of the trapezoid and observe the measures.

At this point, it's probably hard to see any relationships between the area measure and the base and height measurements. Continue sketching to investigate the relationship.

12. Construct the midpoint E of $\overline{DB}$.
13. Mark point E as a center and rotate the entire trapezoid by 180°.
14. Drag parts of the figure and observe the shape formed by the trapezoid and its rotated image.

Double-click point E to mark it as a center. Select the trapezoid. In the Transform menu, choose **Rotate**.



Q1 What shape do the two combined trapezoids form?

Q2 Let b_1 represent the length of base AB and let b_2 represent the length of base CD. What is the length of the base of the shape formed by the combined trapezoids?

Q3 Write a formula for the area of the combined shape in terms of b_1, b_2, and h (for height).

Q4 Write a formula for the area of a single trapezoid in terms of b_1, b_2, and h.

Choose **Calculate** from the Measure menu to open the Calculator. Click once on a measurement to enter it into a calculation. Use parentheses where necessary.

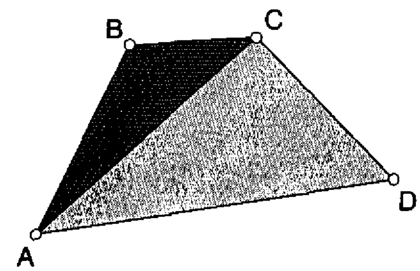
Q5 In your sketch, check that you've derived the correct formula by calculating an expression equal to the area of the trapezoid. Use AB, CD, and the distance from point C to $\overline{AB}$ in your expression. Record your expression below.

Explore More

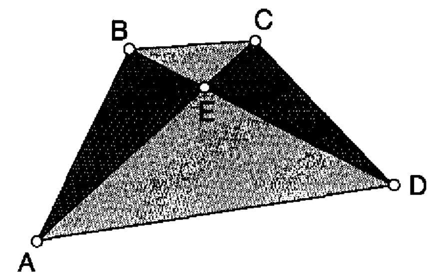
1. Construct the midpoints of the nonparallel sides of the trapezoid. Connect these midpoints with a segment. Use the length of this midsegment to invent a new area formula.
2. Construct a triangle inside your trapezoid whose area is always half the area of the trapezoid. Explain what you did. Is there more than one way to do it?

Dividing Land

Name(s): ____________________



When farmers Clarence and Myrtle died, they left their two daughters their land with instructions to divide it equally. One daughter, Ella, was considerably more conniving than her sister, Jo. The land, unfortunately, was shaped as an irregular quadrilateral, and it wasn't immediately obvious how to divide it equally. Ella first tried to get Jo to agree to split it down the diagonal *AC* shown, with Ella getting region *ACD* and Jo getting region *ABC*. Jo could see that was a bad deal.



Ella then offered to split the land with both diagonals. Ella would take two regions, *AED* and *BEC*, leaving Jo with regions *ABE* and *CED*. This sounded good to Jo, but when she checked it out, she found that the sums of the areas of the respective regions were still not equal. "Ah," said Ella, "but the *products* of the areas of our regions *are* equal!" This stumped Jo, and she agreed to the deal out of sheer awe for Ella's discovery.

Sketch and Investigate

Model this problem with Sketchpad. Is Ella's claim true for all quadrilaterals? Does that mean that this was a fair way to divide the land? Why or why not? See if you can show why Ella's conjecture is true. Write your findings in the space below.

Explore More

1. Come up with a way to divide an irregular quadrilateral using only two segments so that the four regions can be divided equally between two people.

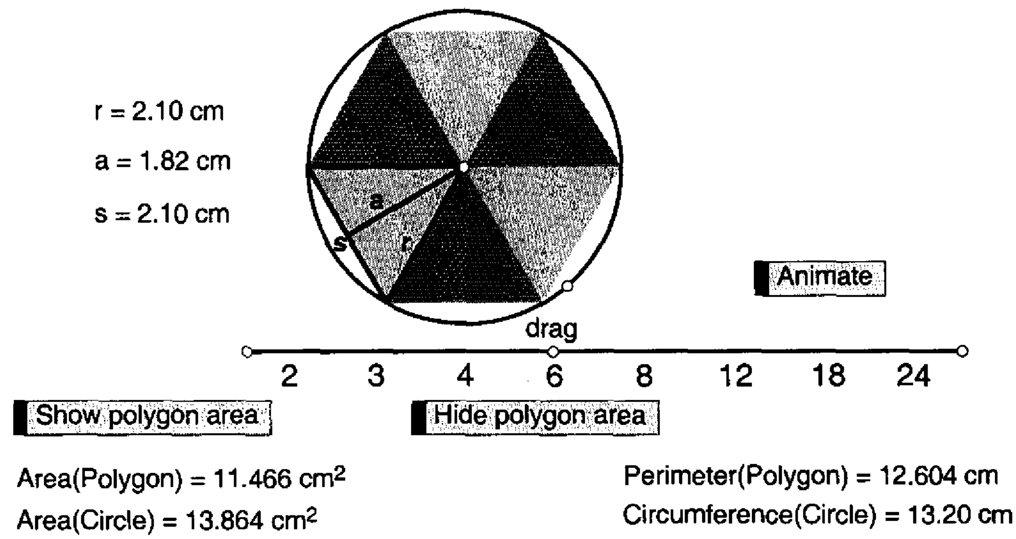
Areas of Regular Polygons and Circles

Name(s): ____________________

A regular polygon has congruent sides and congruent angles. You can divide any regular polygon into congruent triangles by drawing segments from the center to each vertex. You can then use the area of one of these triangles to find the area of the polygon. This method can be extended to derive a formula for the area of a circle. In this activity, you'll manipulate a sketch in which you can vary the number of sides of a regular polygon to see how that affects the area.

Sketch and Investigate

1. Open the sketch **To a Circle.gsp**.



2. Drag (or animate) point "drag" along its segment. Observe how the polygon and the measurements change.
3. Drag to give the polygon four or more sides.

Q1 Each regular polygon with four or more sides is divided into triangles. Look at the triangle with a thick outline. It has segments labeled a for *apothem* and s for side. The side is the base of this triangle and the length of the apothem is the height of the triangle. Write a formula for the area of the triangle using a and s.

Choose **Calculate** from the Measure menu to open the Calculator. Click once on a measurement to enter it into a calculation.

Q2 Use the measurements for a and s to calculate an expression for the area of the polygon. Press the *Show polygon area* button to confirm that you've made the correct expression. Record your expression below. *Note:* Your calculation will disappear if you change the number of sides of the polygon.

Q3 Change the number of sides of the polygon and calculate a new expression for its area. Check this new expression against the area given in the sketch. Record this expression below.

Q4 If you've successfully calculated expressions for a couple polygons, you should be ready to write a general formula for the area of a regular polygon. Write a formula for area A using a for apothem, s for side length, and n for number of sides.

Q5 Write a formula for the perimeter p of a regular polygon with n sides and with side length s.

Q6 Rewrite your formula in Q4 using p instead of s and n.

4. With the polygon area showing, drag or animate point "drag" some more, focusing on what happens to the polygon as the number of sides increases.

Q7 What happens to the polygon as the number of sides increases?

Q8 What does the apothem approach as the number of sides increases?

Q9 What does the perimeter approach as the number of sides increases?

5. Use the measurements for circumference and for r (radius) to calculate an expression equal to the area of the circle.

Q10 Write a formula for the area of a circle A using C for circumference and r for radius.

Q11 The formula for circumference is $C = 2\pi r$. Substitute $2\pi r$ for C in your formula in Q10 and simplify.

New Area Formulas

Name(s): ____________________

Space scientists have discovered a capsule from an extraterrestrial civilization. Within they found mysterious writings that noted intergalactic linguists translated into the area formulas below.

1. To find the area of a triangle, use $A = mh$, where m is the length of the midsegment of the triangle and h is the height of the triangle.
2. To find the area of a trapezoid, use $A = mh$, where m is the length of the midsegment of the triangle and h is the height of the triangle.
3. To find the area of a rhombus, use $A = rs$, where r and s are the lengths of the diagonals.
4. To find the area of a kite, use $A = rs$, where r and s are the lengths of the diagonals.

Sketch and Investigate

Use Sketchpad to determine if all four formulas always work. Construct polygon interiors and measure the areas. Now measure other quantities (heights, midsegments, and so on) and do calculations to test the formulas. Which work? Which don't? Why? Make sure you manipulate your figures to confirm that the formulas that work always work.

In the space below, write explanations for why the formulas do or don't work. Use algebra and what you know about the standard area formulas for these shapes. Correct any formulas that don't work.

Explore More

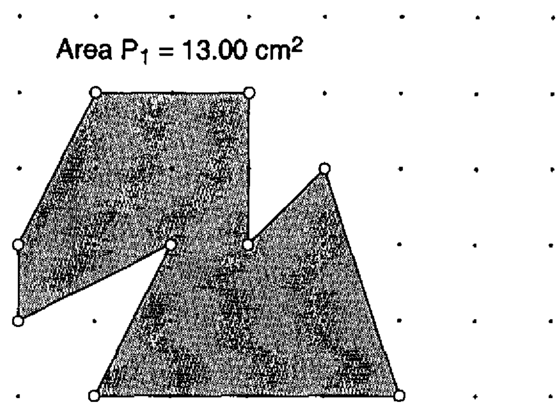
1. See if you can come up with other new area formulas for these or other shapes.

Pick's Theorem

Name(s): ____________________

Pick's theorem is a handy shortcut for finding areas of dot-paper polygons—polygons whose vertices align on a square grid. You will discover Pick's theorem in this activity.

Sketch and Investigate



Polygon p1 has 14 border points and 7 interior points.

1. On the Units panel of Preferences, set the Distance Units to **cm**.

In the Graph menu, choose **Show Grid**, then choose **Snap Points**. Select the grid (by clicking on a grid intersection) and choose **Display | Line Width | Dotted**.

2. Show the grid and turn on point snapping.
3. Hide the axes and the two points. Your screen will now look like a piece of dot paper or a geoboard.

Use the **Segment** tool.

4. Construct a many-sided polygon with vertices on grid points.

Select the polygon's vertices in consecutive order; then, in the Construct menu, choose **Polygon Interior**. While the interior is selected, go to the Measure menu and choose **Area**.

5. Construct the polygon's interior.
6. Measure the polygon's area.

Q1 An interior point is any grid point that lies inside the polygon. A border point is any grid point that lies on the border of the polygon. Some border points are vertices of the polygon, but many polygons have border points that are not vertices.

Count the number of border points b and the number of interior points i in your polygon and record these numbers in the chart below. Then calculate $b/2$ and $(b/2) + i$ and complete the first row of the chart.

	b	$\frac{b}{2}$	i	$\frac{b}{2} + i$	Area
Polygon 1					
Polygon 2					
Polygon 3					
Polygon 4					

Q2 Change the shape of your polygon and find the new values for i, b, and the area. Use these values to complete row 2 of the chart.

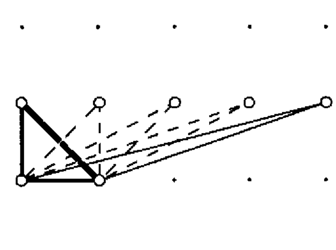
Q3 Continue changing your polygon and recording new entries in your chart until you see a pattern. The pattern you observe is called Pick's theorem. Write Pick's theorem in the space below.

Explore More

1. It is difficult to explain why Pick's theorem works for every type of polygon. It is not as hard to explain why Pick's theorem works for some triangles, since a triangle is one of the simplest of polygons.

   a. Start a new "dot paper" sketch on your computer screen. Make a small triangle with a base of length 1 unit and a height of 1 unit. Explain why Pick's theorem correctly calculates the area of this simple shape.

   b. Without changing the position of the base or the height of the triangle, drag the top vertex horizontally. Explain why changing the shape of the triangle in this way does not change its area. Also explain why Pick's theorem gives the same result for all these different triangles.

   c. Now follow the procedure from parts a and b, using a triangle with a base of 1 unit and a height of 2 units.

   d. Follow the procedure from parts a and b, using a triangle with a base of 2 units and a height of 2 units.

   e. Continue this procedure for some larger triangles.

2. Explain why Pick's theorem works for any rectangle drawn on dot paper.

Squares and Square Roots

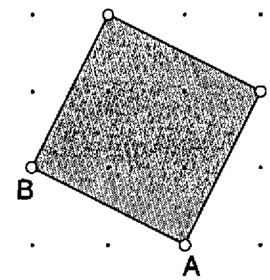
Name(s): ____________________

You can use the Sketchpad grid just like a geoboard or a piece of dot paper. In this activity, you will use squares on a grid to find the square roots of numbers. If you know the area of a square, you can find the square root of its area by measuring one of its sides.

Remember, most square roots are irrational, so many of the values you calculate will be rounded!

Sketch and Investigate

In the Edit menu, choose **Preferences** and go to the Units panel.



1. In Preferences, set the Distance Units to **cm** (centimeters) and the Distance Precision to **thousandths.**

In the Graph menu, choose **Show Grid**, then choose **Snap Points**. Select the grid (by clicking on a grid intersection) and choose **Display | Line Width | Dotted**.

2. Show the grid and turn on point snapping.
3. Hide the axes and the two control points.
4. Use a custom tool to construct a square and its interior by its edge endpoints.

You can use the tool **4/Square (By Edge)** from the sketch **Polygons.gsp**.

5. Measure the area of the square.
6. Measure the length of a side of the square.

Q1 How is a square's side length related to its area?

Select the two measurements. Then, in the Graph menu, choose **Tabulate**.

7. Make a table with these two measurements.
8. Drag a vertex to make the square a different size.

To add an entry, double-click inside the table. If you add an entry that you don't want in your table, choose **Undo** in the Edit menu.

9. Add an entry to your table.
10. Continue adding entries to your square-root table by changing the size of your square.

Q2 Use your table to find the square roots of 12 different numbers less than or equal to 20. Record the numbers and their square roots here.

Square												
Square root												

Q3 Do you think it is possible to find the square root of any whole number using this method? Explain your reasoning.