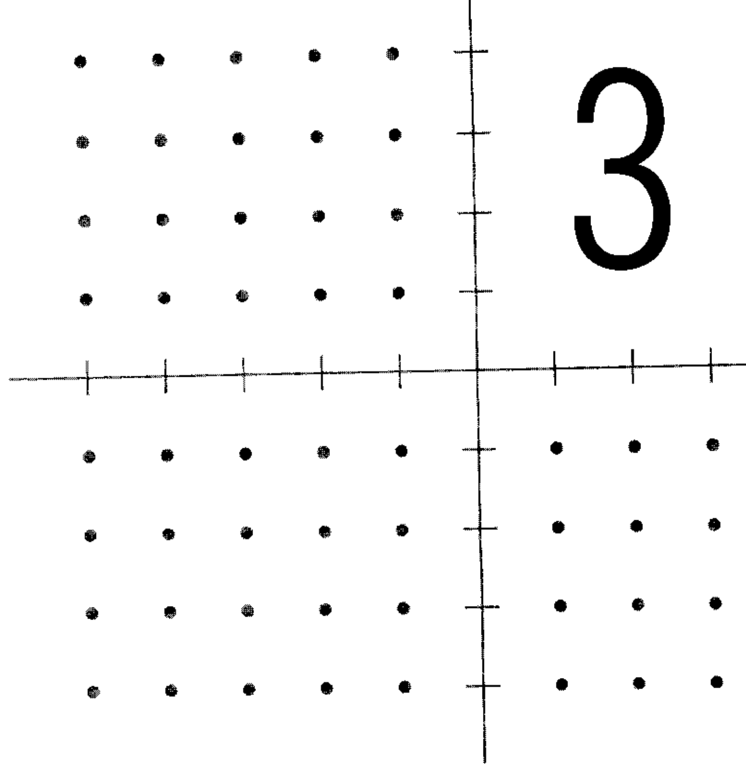
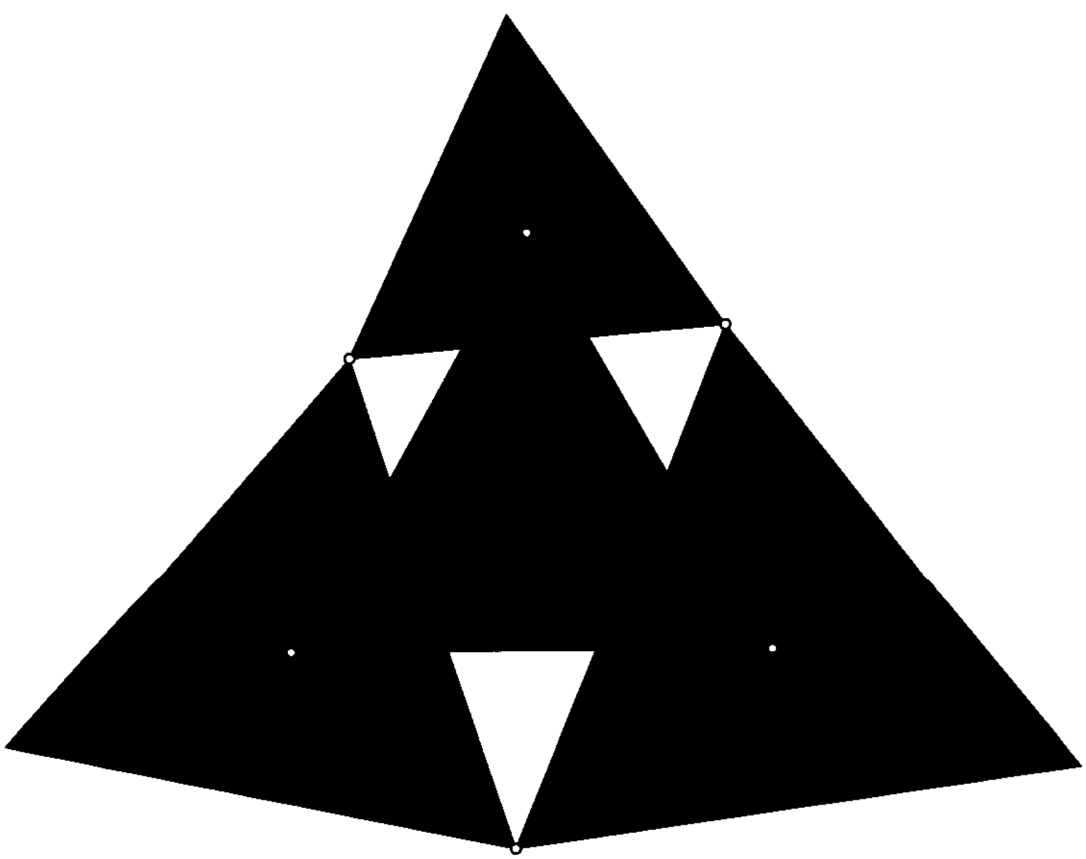


3

Triangles



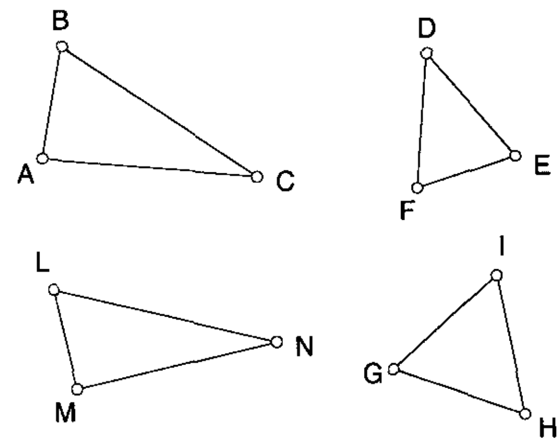
Defining Triangles

Name(s): ____________________

In this lesson, you'll experiment with an ordinary triangle and with special triangles that were constructed with constraints. The constraints limit what you can change in the triangle when you drag so that certain relationships among angles and sides always hold. By observing these relationships, you will classify the triangles.

1. Open the sketch **Classify Triangles.gsp**.
2. Drag different vertices of each of the four triangles to observe and compare how the triangles behave.

Q1 Which of the triangles seems the most flexible? Explain.



Q2 Which of the triangles seems the least flexible? Explain.

To measure each angle, select three points, with the vertex your middle selection. Then, in the Measure menu, choose **Angle**.

3. Measure the three angles in ΔABC.
4. Drag a vertex of ΔABC and observe the changing angle measures. To answer the following questions, note how each angle can change from being acute to being right or obtuse.

Q3 How many acute angles can a triangle have? ____________________

Acute, *right*, and *obtuse* are terms used to classify angles. These terms can also be used to classify triangles according to their angles.

Q4 How many obtuse angles can a triangle have? ____________________

Q5 ΔABC can be an obtuse triangle or an acute triangle. One other triangle in the sketch can also be either acute or obtuse. Which triangle is it? ______________

Q6 Which triangle is always a right triangle, no matter what you drag? ______________

Q7 Which triangle is always an equiangular triangle? ______________

To measure a length, select a segment. Then, in the Measure menu, choose **Length**.

5. Measure the lengths of the three sides of ΔABC.
6. Drag a vertex of ΔABC and observe the changing side lengths.

Scalene, isosceles, and *equilateral* are terms used to classify triangles by relationships among their sides. Because $\triangle ABC$ has no constraints, it can be any type of triangle.

Q8 If none of the side lengths are equal, the triangle is a *scalene* triangle. If two or more of the sides are equal in length, the triangle is *isosceles*. If all three sides are equal in length, it is *equilateral*. Which type of triangle is $\triangle ABC$ most of the time? ____________

Q9 Name a triangle besides $\triangle ABC$ that is scalene most of the time. (Measure if you like, but if you drag things around, you should be able to tell without measuring, just by looking.) ____________

Q10 Which triangle (or triangles) is (are) always isosceles? ____________

Q11 Which triangle is always equilateral? ____________

Q12 For items a–e below, state whether the triangle described is possible or not possible. To check whether each is possible, try manipulating the triangles in the sketch to make one that fits the description. If it's possible to make the triangle, sketch an example on a piece of paper.

a. An obtuse isosceles triangle

b. An acute right triangle

c. An obtuse equiangular triangle

d. An isosceles right triangle

e. An acute scalene triangle

Q13 Write a definition for each of these six terms: *acute triangle, obtuse triangle, right triangle, scalene triangle, isosceles triangle,* and *equilateral triangle*. Use a separate sheet if necessary.

In the Display menu, choose **Show All Hidden** to get an idea of how the triangles in the **Classify Triangles.gsp** sketch were constructed.

Explore More

1. Open a new sketch and see if you can come up with ways to construct a right triangle, an isosceles triangle, and an equilateral triangle. Describe your methods.

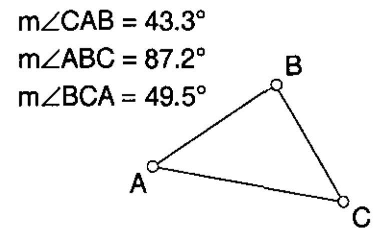
Triangle Sum

Name(s): ____________________

This is a two-part investigation. First you'll investigate and make a conjecture about the sum of the measures of the angles in a triangle, then you'll continue sketching to demonstrate why your conjecture is true.

Sketch and Investigate

1. Construct ΔABC.
2. Measure its three angles.
3. Calculate the sum of the angle measures.
4. Drag a vertex of the triangle and observe the angle sum.



To measure an angle, select three points, with the vertex your middle selection. Then, in the Measure menu, choose **Angle**.

Choose **Calculate** from the Measure menu to open the Calculator. Click once on a measurement to enter it into a calculation.

Q1 What is the sum of the angles in any triangle? ____________

Follow these steps to investigate why your conjecture is true.

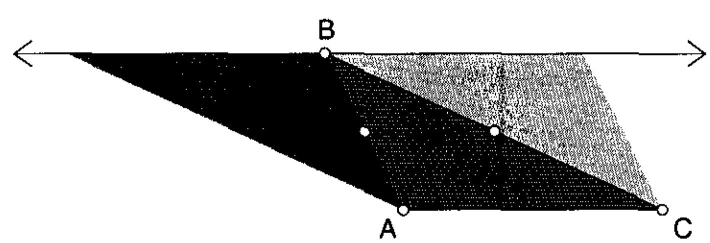
5. Construct a line through point B parallel to $\overline{AC}$.
6. Construct the midpoints of $\overline{AB}$ and $\overline{CB}$.
7. Construct the interior of ΔABC.
8. Mark one of the midpoints as a center for rotation and rotate the interior by 180° about this point.
9. Give the new triangle interior a different color.
10. Mark the other midpoint as a center and rotate the interior by 180° about this point.
11. Give this new triangle interior a different color.

Select point B and $\overline{AC}$; then, in the Construct menu, choose **Parallel Line**.

Select the three vertices; then, in the Construct menu, choose **Triangle Interior**.

Double-click the point to mark it as a center. Select the interior; then, in the Transform menu, choose **Rotate**.

Color is in the Display menu.



12. Drag point B and observe how the three triangles are related to each other and to the parallel line.

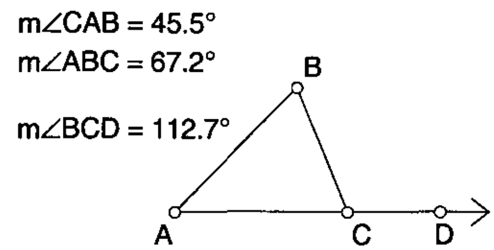
Q2 Explain how each of the three angles at point B is related to one of the three angles in the triangle. Explain how this demonstrates your conjecture from Q1.

Exterior Angles in a Triangle

Name(s): ____________________

An *exterior angle* of a triangle is formed when one of the sides is extended. An exterior angle lies outside the triangle. In this investigation, you'll discover a relationship between an exterior angle and the sum of the measures of the two remote interior angles.

Sketch and Investigate



1. Construct $\triangle ABC$.

Hold down the mouse button on the **Segment** tool and drag right to choose the **Ray** tool.

2. Construct $\overrightarrow{AC}$ to extend side AC.
3. Construct point D on $\overrightarrow{AC}$, outside of the triangle.

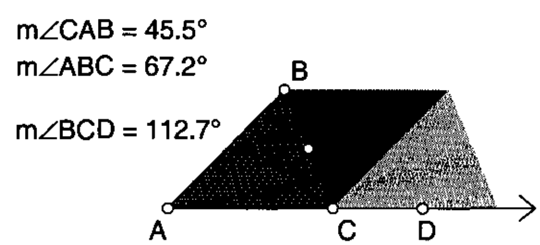
Select, in order, points *B*, *C*, and *D*. Then, in the Measure menu, choose **Angle**.

4. Measure exterior angle BCD.
5. Measure the remote interior angles $\angle ABC$ and $\angle CAB$.
6. Drag parts of the triangle and look for a relationship between the measures of the remote interior angles and the exterior angle.

Choose **Calculate** from the Measure menu to open the Calculator. Click once on a measurement to enter it into a calculation.

Q1 How are the measures of the remote interior angles related to the measure of the exterior angle? Use the Calculator to create an expression that confirms your conjecture.

Follow these steps to see why your conjecture is true.



Select the vertices; then, in the Construct menu, choose **Triangle Interior**.

7. Construct the interior of $\triangle ABC$.

Select point *A* and point *C* in order; then, in the Transform menu, choose **Mark Vector**.

8. Mark AC as a vector.
9. Translate the interior by the marked vector.

Select the interior; then, in the Transform menu, choose **Translate**.

10. Give the new triangle interior a different color.
11. Construct the midpoint of $\overline{BC}$.

Double-click the point to mark it as a center. Select the interior; then, in the Transfrom menu, choose **Rotate**.

12. Mark the midpoint as a center for rotation and rotate the triangle interior about this point by 180°.
13. Give this new triangle interior a different color.

Q2 Explain how the two angles that fill the exterior angle are related to the remote interior angles in the triangle. Explain how this demonstrates your conjecture from Q1. Use the back of this sheet.

Triangle Inequalities

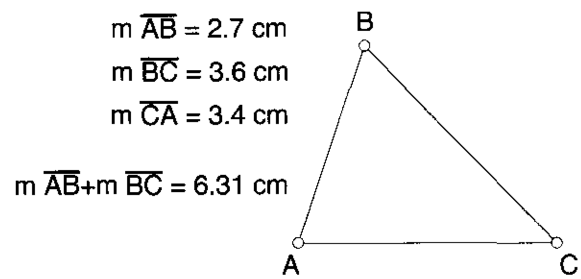
Name(s): ____________________

In this investigation, you'll discover relationships among the measures of the sides and angles in a triangle.

Sketch and Investigate

To measure a side, select it; then, in the Measure menu, choose **Length**.

Choose **Calculate** from the Measure menu to open the Calculator. Click once on a measurement to enter it into a calculation.



1. Construct a triangle.
2. Measure the lengths of the three sides.
3. Calculate the sum of any two side lengths.
4. Drag a vertex of the triangle to try to make the sum you calculated equal to the length of the third side.

Q1 Is it possible for the sum of two side lengths in a triangle to be equal to the third side length? Explain.

Q2 Do you think it's possible for the sum of the lengths of any two sides of a triangle to be less than the length of the third side? Explain.

Q3 Summarize your findings as a conjecture about the sum of the lengths of any two sides of a triangle.

To measure an angle, select three points, with the vertex your middle selection. Then, in the Measure menu, choose **Angle**.

5. Measure $\angle ABC$, $\angle BAC$, and $\angle ACB$.
6. Drag the vertices of your triangle and look for relationships between side lengths and angle measures.

Q4 In each area of the chart below, a longest or shortest side is given. Fill in the chart with the name of the angle with the greatest or least measure, given that longest or shortest side.

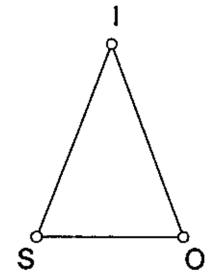
$\overline{AB}$ longest side	Largest angle? _____	$\overline{AC}$ longest side	Largest angle? _____	$\overline{BC}$ longest side	Largest angle? _____
$\overline{AB}$ shortest side	Smallest angle? _____	$\overline{AC}$ shortest side	Smallest angle? _____	$\overline{BC}$ shortest side	Smallest angle? _____

Q5 Summarize your findings from the chart as a conjecture.

Constructing Isosceles Triangles

Name(s): ______________

How many ways can you come up with to construct an isosceles triangle? Try methods that use just the freehand tools (those in the Toolbox) and also methods that use the Construct and Transform menus. Write a brief description of each construction method along with the properties of isosceles triangles that make that method work.



Method 1:

Properties:

Method 2:

Properties:

Method 3:

Properties:

Method 4:

Properties:

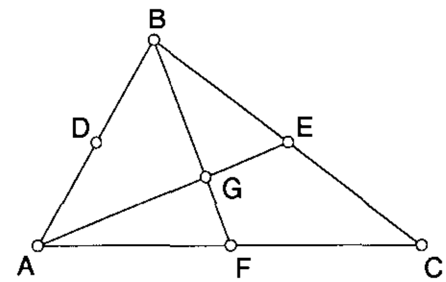
Medians in a Triangle

Name(s): ____________________

A median in a triangle connects a vertex with the midpoint of the opposite side. In previous investigations, you may have discovered properties of angle bisectors, perpendicular bisectors, and altitudes in a triangle. Would you care to make a guess about medians? You may see what's coming, but there are new things to discover about medians, too.

Sketch and Investigate

1. Construct triangle ABC.
2. Construct the midpoints of the three sides.
3. Construct two of the three medians, each connecting a vertex with the midpoint of its opposite side.



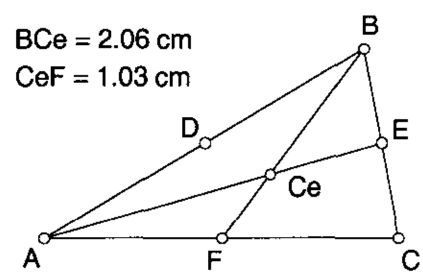
If you've already constructed three medians, select two of them. Then, in the Construct menu, choose **Intersection**.

4. Construct the point of intersection of the two medians.
5. Construct the third median.

Q1 What do you notice about this third median? Drag a vertex of the triangle to confirm that this conjecture holds for any triangle.

Using the **Text** tool, click once on the point to show its label. Double-click the label to change it.

6. The point where the medians intersect is called the *centroid.* Show its label and change it to *Ce* for centroid.



BCe	CeF
2.17 cm	1.08 cm
1.00 cm	0.50 cm
3.24 cm	1.62 cm
2.06 cm	1.03 cm

To measure the distance between two points, select them; then, in the Measure menu, choose **Distance**.

7. Measure the distance from B to Ce and the distance from Ce to the midpoint F.
8. Drag vertices of $\triangle ABC$ and look for a relationship between BCe and CeF.

Select the two measurements; then, in the Graph menu, choose **Tabulate**.

9. Make a table with these two measures.
10. Change the triangle and double-click the table values to add another entry.
11. Keep changing the triangle and adding entries to your table until you can see a relationship between the distances BCe and CeF.

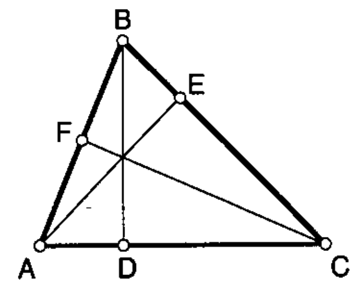
Altitudes in a Triangle (continued)

Select everything in your sketch. Then, press the **Custom** tools icon (the bottom tool in the Toolbox) and choose **Create New Tool** from the menu that appears.

Starting with three points (the tool's "givens"), the Altitude tool will construct a triangle and an altitude from one of the vertices. To use the tool, click on the **Custom** tools icon to select the most recently created tool, then click on the three triangle vertices.

8. Drag your triangle so that it is acute again (with the altitude falling inside the triangle).
9. Make a custom tool for this construction. Name the tool "Altitude."
10. Use your custom tool on the triangle's vertices to construct a second altitude. Don't worry if you accidentally construct the altitude that already exists. Just use the tool on the vertices again in a different order until you get another altitude.
11. Use your Altitude tool to construct the third altitude in the triangle.
12. Drag the triangle and observe how the three altitudes behave.

Q3 What do you notice about the three altitudes when the triangle is acute?



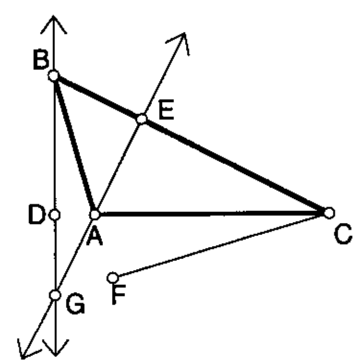
Step 11

Q4 What do you notice about the altitudes when the triangle is obtuse?

When the triangle is obtuse, the three altitudes don't intersect. But do you think they would if they were long enough? Follow the steps below to investigate that question.

If you've already constructed three lines, select two of them; then, in the Construct menu, choose **Intersection**.

13. Make sure the triangle is obtuse. Construct two lines that each contain an altitude.
14. Construct their point of intersection. This point is called the *orthocenter* of the triangle.
15. Construct a line containing the third altitude.
16. Drag the triangle and observe the lines.



Steps 13 and 14

Q5 What do you notice about the lines containing the altitudes?

Explore More

1. Hide everything in your sketch except the triangle and the orthocenter. Make and save a custom tool called "Orthocenter." You can use this tool in other investigations of triangle centers.

Angle Bisectors in a Triangle

Name(s): ____________________

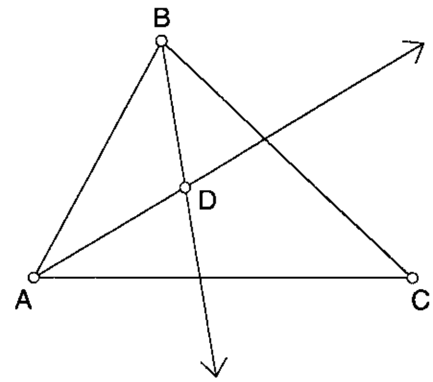
In this investigation, you'll discover some properties of angle bisectors in a triangle.

Sketch and Investigate

Select three points, with the vertex your middle selection. Then, in the Construct menu, choose **Angle Bisector**.

Click at the intersection with the **Arrow** or the **Point** tool. Or select the two bisectors, then, in the Construct menu, choose **Intersection**.

1. Construct triangle *ABC*.
2. Construct the bisectors of two of the three angles: $\angle A$ and $\angle B$.
3. Construct point *D*, the point of intersection of the two angle bisectors.
4. Construct the bisector of $\angle C$.



Steps 1–3

Q1 What do you notice about this third angle bisector (not shown)? Drag each vertex of the triangle to confirm that this observation holds for any triangle.

Select point *D* and one side of the triangle. Then, in the Measure menu, choose **Distance**. Repeat for the other two sides.

5. Measure the distances from *D* to each of the three sides.
6. Drag each vertex of the triangle and observe the distances.

Q2 The point of intersection of the angle bisectors in a triangle is called the *incenter*. Write a conjecture about the distances from the incenter of a triangle to the three sides.

Explore More

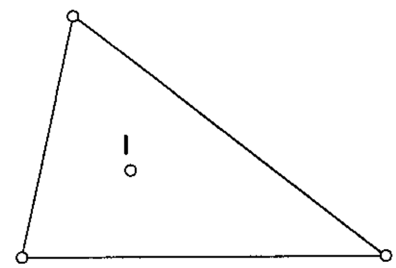
1. An inscribed circle is a circle inside a triangle that touches each of the three sides at one point. Construct an inscribed circle that stays inscribed no matter how you drag the triangle. (*Hint:* You'll need to construct a perpendicular line.)
2. Make and save a custom tool for constructing the incenter of a triangle (with or without the inscribed circle). You can use this tool when you investigate properties of other triangle centers.
3. Explain why the intersection of the angle bisectors would be the center of the inscribed circle. *Hint:* Recall that any point on an angle bisector is equidistant from the two sides of the angle. Why would the incenter be equidistant from the three sides of the triangle?

The Euler Segment

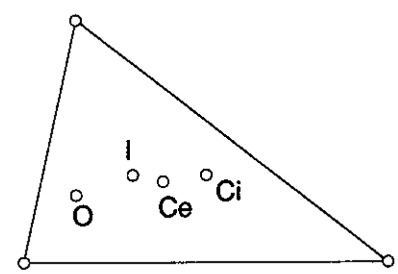
Name(s): ______________________________

In this investigation, you'll look for a relationship among four points of concurrency: the incenter, the circumcenter, the orthocenter, and the centroid. You'll use custom tools to construct these triangle centers, either those you made in previous investigations or pre-made tools.

Sketch and Investigate



1. Open a sketch (or sketches) of yours that contains tools for the triangle centers: incenter, circumcenter, orthocenter, and centroid. Or open **Triangle Centers.gsp**.
2. Construct a triangle.
3. Use the **Incenter** tool on the triangle's vertices to construct its incenter.
4. If necessary, give the incenter a label that identifies it, such as *I* for incenter.
5. You need only the triangle and the incenter for now, so hide anything extra that your custom tool may have constructed (such as angle bisectors or the incircle).



6. Use the **Circumcenter** tool on the same triangle. Hide any extras so that you have just the triangle, its incenter, and its circumcenter. If necessary, give the circumcenter a label that identifies it.
7. Use the **Orthocenter** tool on the same triangle, hide any extras, and label the orthocenter.
8. Use the **Centroid** tool on the same triangle, hide extras, and label the centroid. You should now have a triangle and the four triangle centers.

Q1 Drag your triangle around and observe how the points behave. Three of the four points are always collinear. Which three?

9. Construct a segment that contains the three collinear points. This is called the *Euler segment.*

Q2 Drag the triangle again and look for interesting relationships on the Euler segment. Be sure to check special triangles, such as isosceles and right triangles. Describe any special triangles in which the triangle centers are related in interesting ways or located in interesting places.

Q3 Which of the three points are always endpoints of the Euler segment and which point is always between them?

To measure the distance between two points, select the two points. Then, in the Measure menu, choose **Distance**. (Measuring the distance between points is an easy way to measure the length of part of a segment.)

10. Measure the distances along the two parts of the Euler segment.

Q4 Drag the triangle and look for a relationship between these lengths. How are the lengths of the two parts of the Euler segment related? Test your conjecture using the Calculator.

Explore More

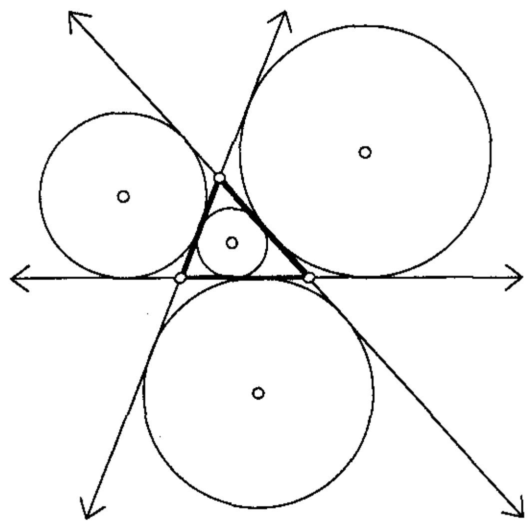
1. Construct a circle centered at the midpoint of the Euler segment and passing through the midpoint of one of the sides of the triangle. This circle is called the *nine-point circle.* The midpoint it passes through is one of the nine points. What are the other eight? (*Hint:* Six of them have to do with the altitudes and the orthocenter.)
2. Once you've constructed the nine-point circle, drag your triangle around and investigate special triangles. Describe any triangles in which some of the nine points coincide.

Excircles of a Triangle

Name(s): ______________________

You may have learned previously to inscribe a circle inside a triangle: Find the intersection of the angle bisectors and use that point as the center of the circle. In this demonstration, you'll define an *excircle* and investigate its properties.

Sketch and Investigate

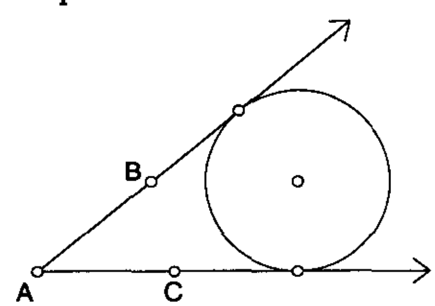


1. Open the sketch **Excircles.gsp**. You'll see a small triangle with a small circle inside it (the incircle) and three larger circles outside of it (excircles).
2. Drag any vertex of the small triangle.

Q1 What do you notice about the incircle?

Q2 What do you notice about each excircle?

To better understand what an excircle is, you should experiment with constructing circles inside angles. Follow the steps below.



3. Open a new sketch, but leave the excircles sketch open.
4. Use the **Ray** tool to construct an angle *BAC*, as shown.
5. Experiment to discover a way to construct a circle that is tangent to both sides of the angle.

Press and hold down the mouse button on the **Segment** tool to show the **Straightedge** palette. Drag right to choose the **Ray** tool.

Q3 Where must the center of the circle be located?

6. Return to the excircles sketch and drag things around, keeping in mind what you may have just discovered about circles and angles.

7. Draw a ray from one vertex of the triangle through the center of any of the excircles.

Q4 What is special about this ray? How do you know?

8. Draw rays from each vertex of the triangle through each excircle center.

9. Drag the triangle vertices and observe relationships in the figure.

Q5 Write as many conjectures as you can about relationships in this sketch.

Explore More

1. See if you can re-create this sketch from scratch. For hints, try showing everything that is hidden.

The Surfer and the Spotter

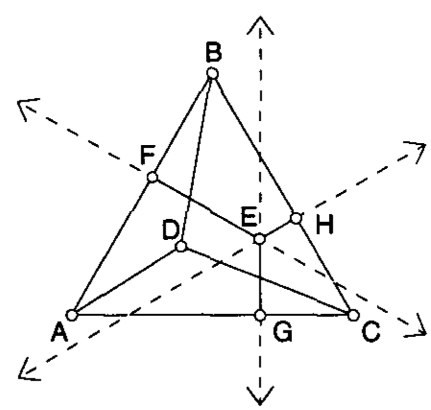
Name(s): ____________________

Two shipwreck survivors manage to swim to a desert island. As it happens, the shape of the island is a perfect equilateral triangle. The survivors have very different dispositions. Sarah soon discovers that the surfing is outstanding on all three of the island's coasts, so she crafts a surfboard from a fallen tree. Because there is plenty of food on the island, she's content to stay on the island and surf for the rest of her days. Spencer, on the other hand, is more a social animal and sorely misses civilization. Every day he goes to a different corner of the island and searches the waters for passing ships. Each castaway wants to locate a home in the place that best suits his or her needs. They have no interest in living in the same place, though if it turns out to be advantageous, neither is against the idea either. Sarah wants to visit each beach with equal frequency, so she wants to find the spot that minimizes the total length of the three paths from home to the three sides of the island. Spencer wants his house to be situated so that the total length of the three paths from his home to the three corners of the island is minimized. Where should they locate their huts?

Sketch and Investigate

Use a custom tool or construct the triangle from scratch.

1. Construct an equilateral triangle ABC.



2. Construct $\overline{DA}$, $\overline{DB}$, and $\overline{DC}$, where D is any point inside the triangle. Point D represents Spencer's hut and the segments represent paths to the corners of the island.

3. Construct point E anywhere inside the triangle.

Select point E and the three sides. Then, in the Construct menu, choose **Perpendicular Lines**. You'll get all three perpendicular lines.

4. Construct lines through point E perpendicular to each of the three sides of the triangle.

5. Construct $\overline{EF}$, $\overline{EG}$, and $\overline{EH}$, where F, G, and H are the points where the perpendiculars intersect the sides of the triangle.

6. Hide the perpendicular lines. Point E represents Sarah's hut and the segments from it represent the paths from her hut to the beaches.

Choose **Calculate** from the Measure menu to open the Calculator. Click once on a measurement to enter it into a calculation.

7. Measure DA, DB, and DC and calculate $DA + DB + DC$.

8. Measure EF, EG, and EH and calculate $EF + EG + EH$.

9. Move points D and E (Spencer and Sarah) around inside your triangle. See if you can find the best location for each castaway.

Q1 What are the best locations for Spencer's and Sarah's huts? On a separate sheet, explain why these are the best locations.

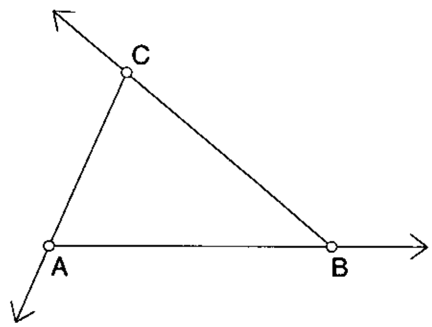
Morley's Theorem

Name(s): ____________________

You may know that it's impossible to trisect an angle with compass and straightedge. Sketchpad, however, makes it easy to trisect an angle. In this investigation, you'll trisect the three angles in a triangle and discover a surprising fact about the intersections of these angle trisections.

Sketch and Investigate

1. Use the **Ray** tool to construct triangle *ABC*, drawing your rays in counterclockwise order as shown.



Step 1

> Choose **Calculate** from the Measure menu to open the Calculator. Click once on a measurement to enter it into a calculation.

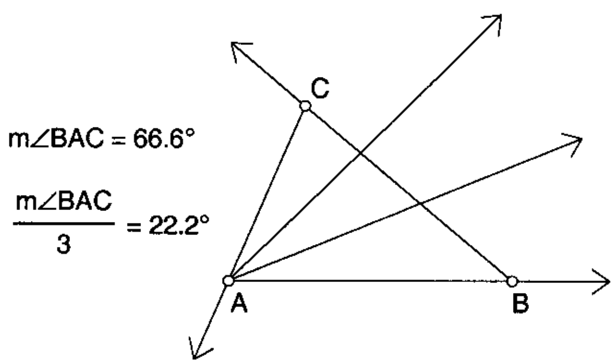
2. Measure $\angle BAC$.
3. Use the Calculator to create an expression for $m\angle BAC/3$.

> Double-click point *A* to mark it as a center.

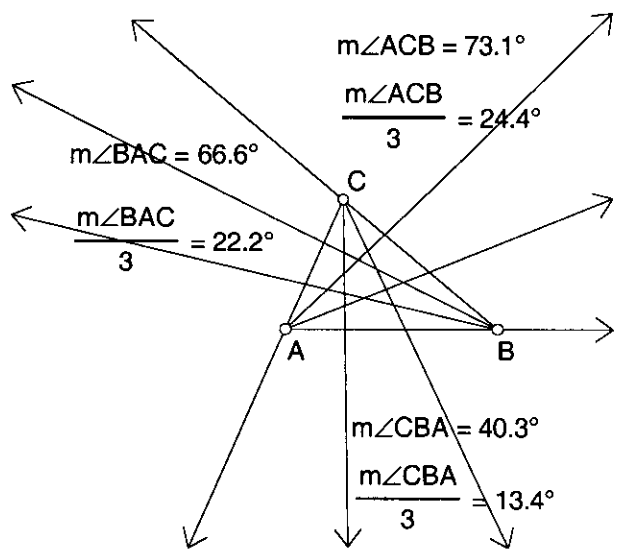
4. Mark point *A* as a center for rotation.

> Select the calculation; then, in the Transform menu, choose **Mark Angle Measurement**.

5. Mark the angle measurement $m\angle BAC/3$.
6. Rotate ray *AB* by the marked angle. Rotate it again to trisect $\angle CAB$.



Steps 2–6



Step 7

7. Measure the other two angles and repeat steps 3 through 6 on those angles to trisect them.
8. Morley's theorem states that certain intersections of these angle trisectors form an equilateral triangle. Can you find it? Drag vertices of the triangle and watch the intersections of the trisectors.

Construct the intersection points and select them. Then, in the Construct menu, choose **Triangle Interior**.

9. When you think you know which intersections form an equilateral triangle, construct those intersection points and the equilateral triangle's interior.
10. Drag to confirm that you've constructed the triangle at the correct intersections. If you can't tell for sure by looking, make the measurements necessary to confirm that the triangle is equilateral.

Q1 State Morley's theorem.

Explore More

1. See if you can find other relationships or special triangles in your figure.
2. Construct rays from the vertices of your original triangle through the opposite vertices of the equilateral triangle. What do you notice?
3. Construct a triangle using lines instead of rays. Trisect one set of exterior angles. Can you find an equilateral triangle among the intersections of these trisectors?

Napoleon's Theorem

Name(s): ______________________

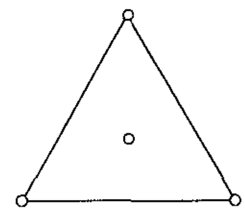
French emperor Napoleon Bonaparte fancied himself as something of an amateur geometer and liked to hang out with mathematicians. The theorem you'll investigate in this activity is attributed to him.

Sketch and Investigate

1. Construct an equilateral triangle. You can use a pre-made custom tool or construct the triangle from scratch.

One way to construct the center is to construct two medians and their point of intersection.

2. Construct the center of the triangle.
3. Hide anything extra you may have constructed to construct the triangle and its center so that you're left with a figure like the one shown at right.

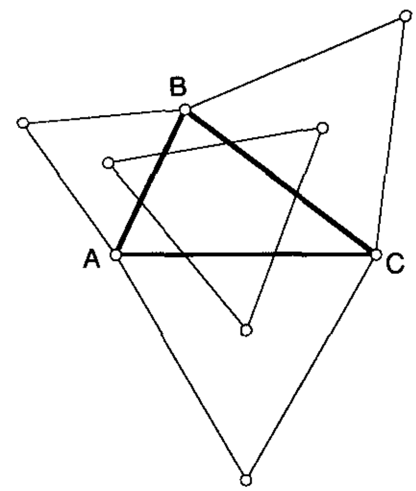


Select the entire figure; then choose **Create New Tool** from the Custom Tools menu in the Toolbox (the bottom tool).

4. Make a custom tool for this construction.

Next, you'll use your custom tool to construct equilateral triangles on the sides of an arbitrary triangle.

5. Open a new sketch.
6. Construct ΔABC.



Be sure to attach each equilateral triangle to a pair of triangle *ABC*'s vertices. If your equilateral triangle goes the wrong way (overlaps the interior of ΔABC) or is not attached properly, undo and try attaching it again.

7. Use the custom tool to construct equilateral triangles on each side of ΔABC.
8. Drag to make sure each equilateral triangle is stuck to a side.
9. Construct segments connecting the centers of the equilateral triangles.
10. Drag the vertices of the original triangle and observe the triangle formed by the centers of the equilateral triangles. This triangle is called the outer Napoleon triangle of ΔABC.

Q1 State what you think Napoleon's theorem might be.

Explore More

1. Construct segments connecting each vertex of your original triangle with the most remote vertex of the equilateral triangle on the opposite side. What can you say about these three segments?