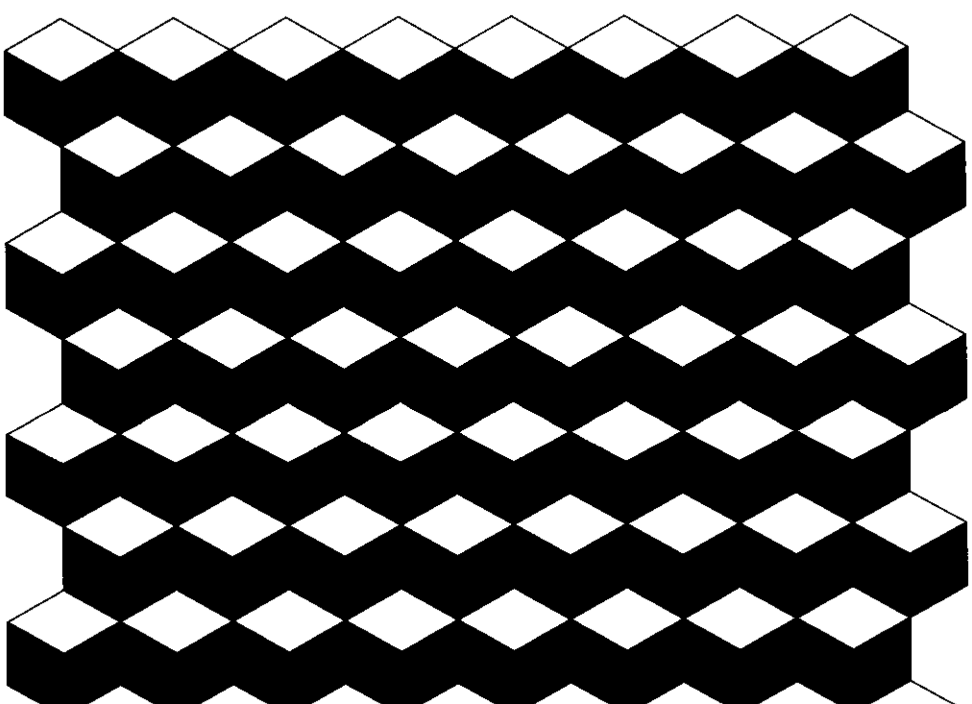
2

Transformations, Symmetry, and Tessellations



Introducing Transformations

Name(s): ____________________

A *transformation* is a way of moving or changing a figure. There are three types of basic transformations that preserve the size and shape of the figure. These three, *reflections*, *rotations*, and *translations*, are called *isometries*. Isometries flip, turn, or slide a figure but never bend or distort it. In this activity, you'll experiment with basic isometries by transforming a flag-shaped polygon. (You'll use this shape because it's easy to keep track of where it's pointing.)

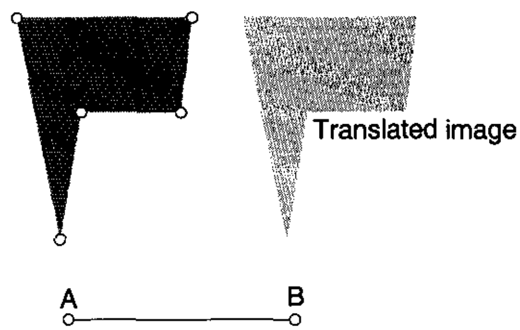
Sketch and Investigate: Translations

Hold down the Shift key as you create points so they'll stay selected as you create new ones. Then choose **Pentagon Interior** from the Construct menu.

1. Construct the vertices of a flag shape and construct its interior.

A brief animation indicates that you've marked a vector. Nothing else will happen in your sketch yet.

2. In order to translate a shape, you need to indicate a direction and a distance. To do this, construct segment AB. Then select, in order, point A and point B. In the Transform menu, choose **Mark Vector.**



3. Select the interior of the flag; then, in the Transform menu, choose **Translate**. Make sure By Marked Vector is checked in the Translate dialog box, then click Translate.

With the interior selected, choose a color from the Color submenu of the Display menu.

4. Change the color of the translated image.

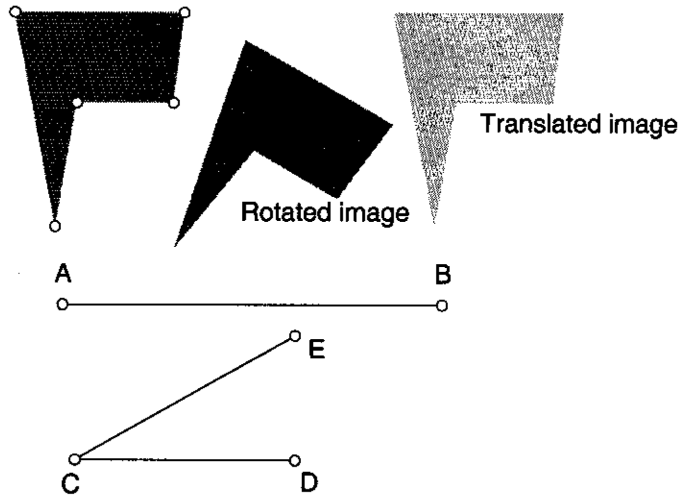
Double-click the polygon interior with the **Text** tool to edit its label.

5. Label the translated polygon *Translated image.*
6. Drag point B to change your vector, and observe the relationship between the translated image and the original figure.

Q1 Compare the translated image to the original figure. How are they different and how are they the same?

Sketch and Investigate: Rotations

7. In order to rotate a shape, you need to indicate a center of rotation and an angle of rotation. Start by creating angle *ECD* using two attached segments, as indicated at right.



Double-click the point to mark it as a center.

8. Mark point *C* as a center of rotation.

Select, in order, points *E*, *C*, and *D*. Then choose **Mark Angle** from the Transform menu.

9. Mark ∠*ECD* as an angle of rotation.

Select the interior. Then choose **Rotate** from the Transform menu.

10. Rotate the original flag-shaped interior by the marked angle.
11. Change the color of the rotated image and label it *Rotated image*.
12. Drag point *D* to change your angle, and observe the relationship between the rotated image and the original figure.

Q2 Compare the rotated image to the original figure. How are they different and how are they the same?

Sketch and Investigate: Reflections

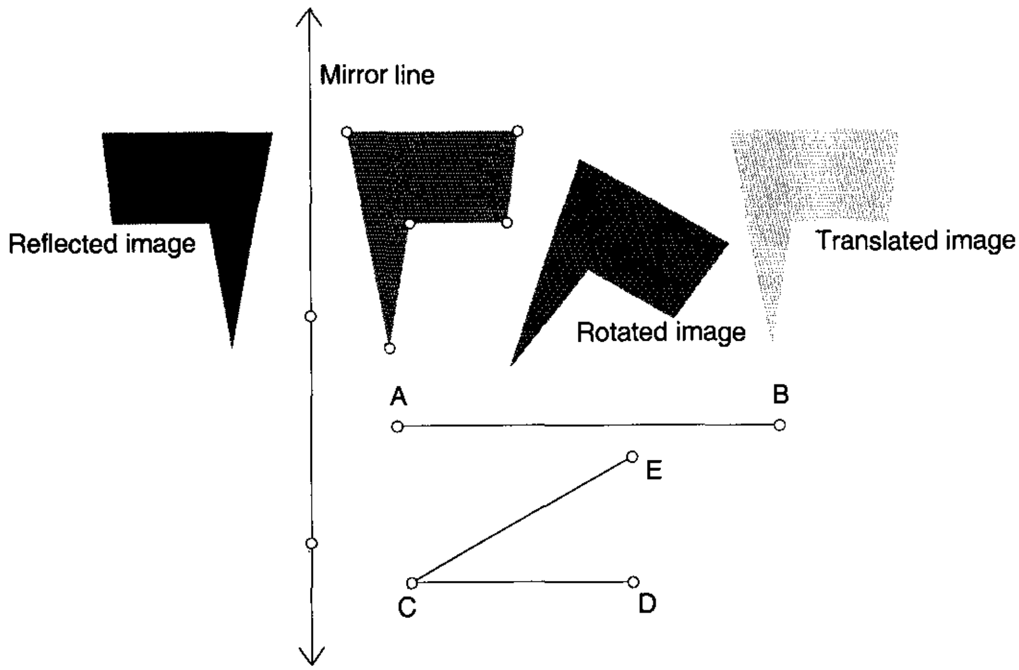
13. To reflect a shape, you need a *mirror line* (also called a *line of reflection*). Draw a line and label it *Mirror line*.

Double-click the line to mark it as a mirror.

14. Mark the line as a mirror.

Select the interior and choose **Reflect** from—you guessed it—the Transform menu.

15. Reflect the original flag-shaped interior. Your image may end up off the screen. If so, move the original figure closer to the mirror line.
16. Change the color of the reflected image. Label it *Reflected image*.
17. Drag your mirror line, and observe the relationship between the reflected image and the original figure.



Q3 Compare the reflected image to the original figure. How are they different and how are they the same?

Q4 Explain whether it is possible for any of the three images in your sketch to lie directly on top of one another. Experiment by dragging different parts of your sketch.

Explore More

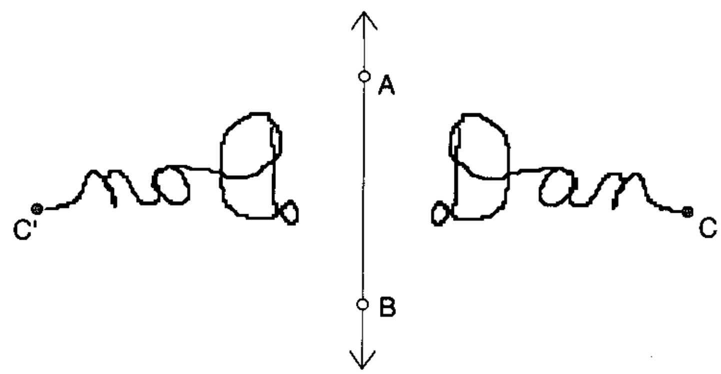
1. Use reflections, rotations, translations, or combinations of these transformations to make a design.
2. Reflect a figure over a line, then reflect the image over a second line that intersects the first. What single transformation would take your original figure to the second reflected image?
3. Reflect a figure over a line, then reflect the image over a second line that is parallel to the first. What single transformation is the same as this combination of two reflections?

Properties of Reflection

Name(s): ____________________

When you look at yourself in a mirror, how far away does your image in the mirror appear to be? Why is it that your reflection looks just like you, but backward? Reflections in geometry have some of the same properties of reflections you observe in a mirror. In this activity, you'll investigate the properties of reflections that make a reflection the "mirror image" of the original.

Sketch and Investigate: Mirror Writing



1. Construct vertical line *AB*.
2. Construct point *C* to the right of the line.
3. Mark $\overleftrightarrow{AB}$ as a mirror.

   Double-click on the line.

4. Reflect point *C* to construct point *C*′.
5. Turn on **Trace Points** for points *C* and *C*′.

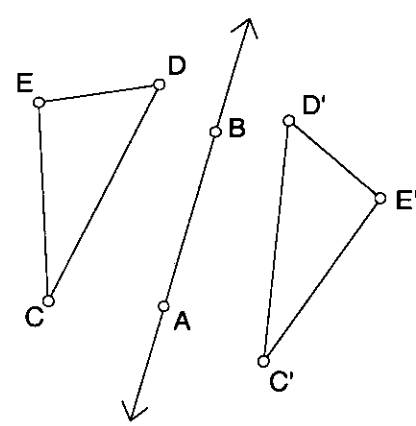
   Select the two points; then, in the Display menu, choose **Trace Points**. A check mark indicates that the command is turned on. Choose **Erase Traces** when you wish to erase your traces.

6. Drag point *C* so that it traces out your name.

Q1 What does point *C*′ trace?

7. For a real challenge, try dragging point *C*′ so that point *C* traces out your name.

Sketch and Investigate: Reflecting Geometric Figures



8. Turn off **Trace Points** for points *C* and *C*′.

   Select points *C* and *C*′. In the Display menu, you'll see **Trace Points** checked. Choose it to uncheck it.

9. In the Display menu, choose **Erase Traces.**
10. Construct $\triangle CDE$.
11. Reflect $\triangle CDE$ (sides and vertices) over $\overleftrightarrow{AB}$.

    Select the entire figure; then, in the Transform menu, choose **Reflect**.

12. Drag different parts of either triangle and observe how the triangles are related. Also drag the mirror line.

13. Measure the lengths of the sides of triangles *CDE* and *C′D′E′*.

Select three points that name the angle, with the vertex your middle selection. Then, in the Measure menu, choose **Angle**.

14. Measure one angle in ΔCDE and measure the corresponding angle in $\Delta C'D'E'$.

Q2 What effect does reflection have on lengths and angle measures?

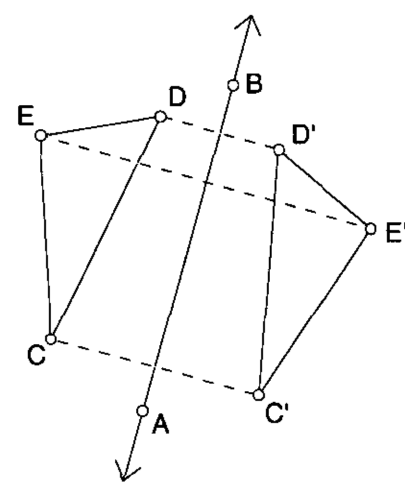
Q3 Are a figure and its mirror image always congruent? State your answer as a conjecture.

Your answer to Q4 demonstrates that a reflection reverses the *orientation* of a figure.

Q4 Going alphabetically from *C* to *D* to *E* in ΔCDE, are the vertices oriented in a clockwise or counter-clockwise direction? In what direction (clockwise or counter-clockwise) are vertices *C′*, *D′*, and *E′* oriented in the reflected triangle?

Line Width is in the Display menu.

15. Construct segments connecting each point and its image: *C* to *C′*, *D* to *D′*, and *E* to *E′*. Make these segments dashed.



You may wish to construct points of intersection and measure distances to look for relationships between the mirror line and the dashed segments.

16. Drag different parts of the sketch around and observe relationships between the dashed segments and the mirror line.

Q5 How is the mirror line related to a segment connecting a point and its reflected image?

Explore More

1. Suppose Sketchpad didn't have a Transform menu. How could you construct a given point's mirror image over a given line? Try it. Start with a point and a line. Come up with a construction for the reflection of the point over the line using just the tools and the Construct menu. Describe your method.
2. Use a reflection to construct an isosceles triangle. Explain what you did.

Reflections in the Coordinate Plane

Name(s): ____________

In this activity, you'll investigate what happens to the coordinates of points when you reflect them across the *x*- and *y*-axes in the coordinate plane.

In the Graph menu, first choose **Show Grid**, then choose **Snap Points**.

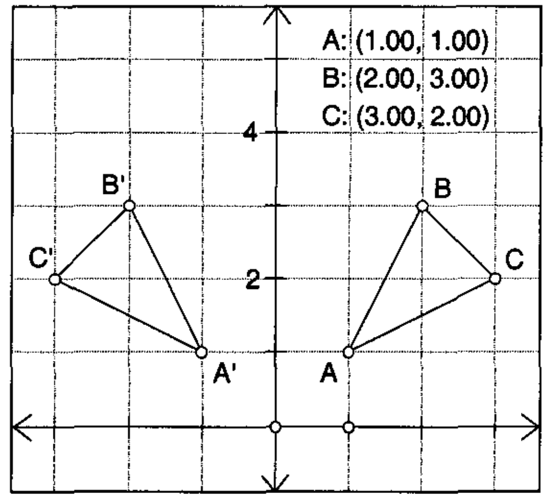
1. Show the grid and turn on point snapping.
2. Draw ΔCDE with vertices on the grid.
3. Measure the coordinates of each vertex.

Double-click the axis to mark it as a mirror.

4. Mark the *y*-axis as a mirror.

Select the entire figure; then, in the Transform menu, choose **Reflect**.

5. Reflect the triangle.
6. Measure the coordinates of the image's vertices.

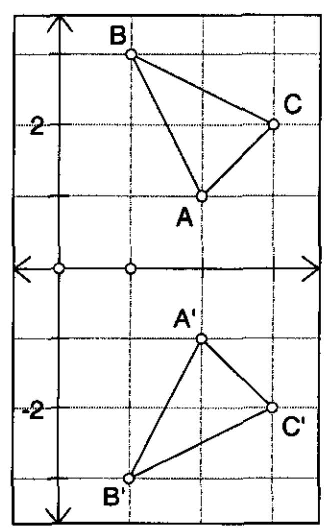


7. Drag vertices to different points on the grid and look for a relationship between a point's coordinates and the coordinates of the reflected image across the *y*-axis.

Q1 Describe any relationship you observe between the coordinates of the vertices of your original triangle and the coordinates of their reflected images across the *y*-axis.

8. Now mark the *x*-axis as a mirror and reflect your original triangle.
9. Before you measure coordinates, can you guess what they'll be? Measure to confirm.

Q2 Describe any relationship you observe between the coordinates of the original points and the coordinates of their reflected images across the *x*-axis.



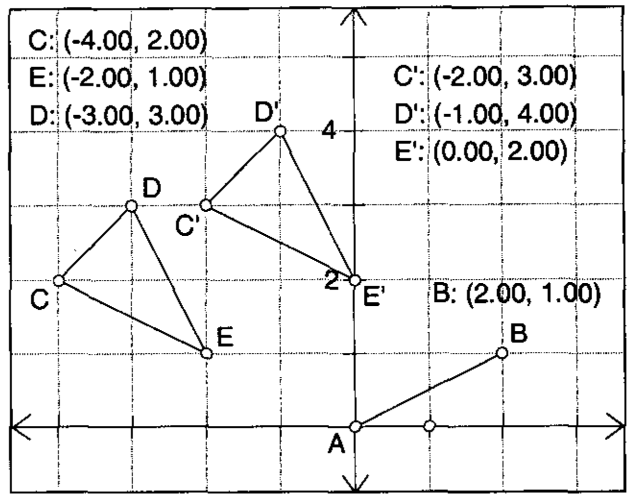
Explore More

1. Draw a line on the grid that passes through the origin and makes a 45° angle with the *x*-axis (in other words, the line $y = x$). Reflect your triangle across this line. What do you notice about the coordinates of the vertices of this image?

Translations in the Coordinate Plane

Name(s): ______________

In this activity, you'll investigate what happens to the coordinates of points when they're translated in the coordinate plane.



In the Graph menu, first choose **Show Grid**, then choose **Snap Points**.

1. Show the grid and turn on point snapping.

Using the **Text** tool, click on a point to display its label. Double-click the label to change it.

2. Draw a segment from the origin to anywhere on the grid. Label the origin *A* and the other endpoint *B*.

Select point *A* and point *B*, in that order; then, in the Transform menu, choose **Mark Vector**. Watch for the animation indicating the marked vector.

3. Measure the coordinates of point *B*.
4. Mark vector *AB*.
5. Draw ΔCDE with vertices on the grid.

Select the entire figure; then, in the Transform menu, choose **Translate**.

6. Translate the triangle by the marked vector.
7. Measure the coordinates of the two triangles' six vertices.
8. Experiment by dragging point *B* or any of the triangle vertices. Look for a relationship between a point's coordinates and the coordinates of its image under a translation.

Q1 Where can you drag point *B* so that the original points and the corresponding image points always have the same *y*-coordinates but have different *x*-coordinates?

Q2 Where can you drag point *B* so that the original points and the corresponding image points always have the same *x*-coordinates but have different *y*-coordinates?

Q3 When the vector defined by the origin and point *B* translates your original triangle to the left and up, what must be true of the coordinates of point *B*?

Q4 Suppose point *B* has coordinates (a, b). What are the coordinates of the image of a point (x, y) under a translation by (a, b)?

The Burning Tent Problem

Name(s): ____________________

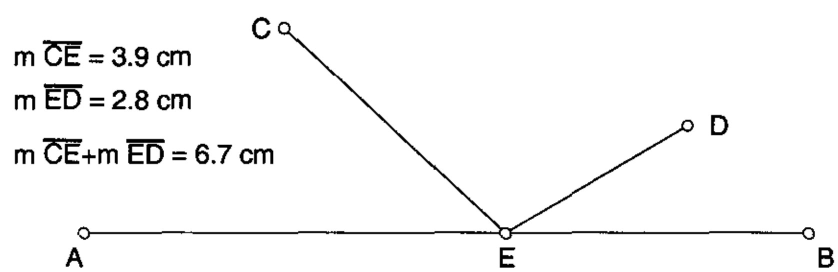
A camper out for a hike is returning to her campsite. The shortest distance between her and her campsite is along a straight line, but as she approaches her campsite, she sees that her tent is on fire! She must run to the river to fill her canteen, then run to her tent to put out the fire. What's the shortest path she can take? The answer to this question is related to the path a pool ball takes when it bounces off a cushion and the path a ray of light takes when it bounces off a mirror. In this activity, you'll investigate the minimal two-part path that goes from a point to a line and then to another point.

Sketch and Investigate

1. Construct a long horizontal segment *AB* to represent the river.
2. Construct points *C* and *D* on the same side of the segment. Point *C* represents the camper and point *D* represents the tent.
3. Construct $\overline{CE}$ and $\overline{ED}$, where point *E* is any point on $\overline{AB}$. These segments together show a path the camper might take running to the river and then to her tent.
4. Measure the lengths of $\overline{CE}$ and $\overline{ED}$.

Choose **Calculate** from the Measure menu to open the Calculator. Click once on a measurement in your sketch to enter it into a calculation. →

5. Use the Calculator to find the sum of these two lengths.



To precisely locate point *E*, you might need to change your precision for calculations to thousandths. To do this, in the Edit menu, choose **Preferences**. →

6. Drag point *E* and watch the calculated sum change. Move point *E* to the location on the river to which the camper should run.

Select three points that name the angle, with the vertex your middle selection. Then, in the Measure menu, choose **Angle**. →

7. Measure the incoming and outgoing angles the camper's path makes with the river (angles *CEA* and *DEB*).

Q1 Once you've found the minimal path, what appears to be true about the incoming angle and the outgoing angle? (See if you can use the angle measures to make the distance sum even shorter.)

So far, you've dragged point E to find an approximate minimal path. Next, you'll discover how to construct such a path.

Double-click the segment.

8. Mark segment AB as a mirror. (Now that it's marked as a mirror, you can stop thinking of it as a river.)

Select the point; then, in the Transform menu, choose **Reflect**.

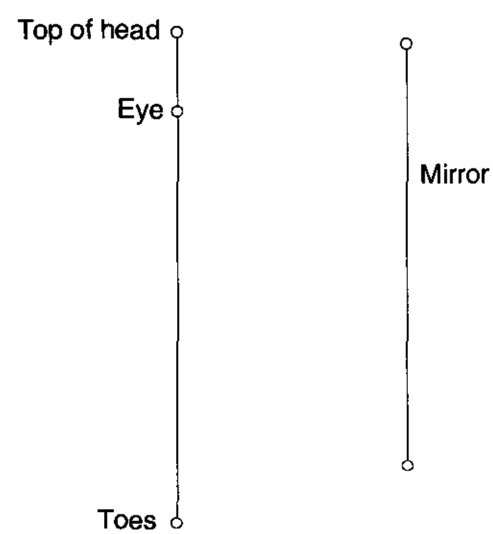
9. Reflect point D across this segment to create point D'.
10. Construct $\overline{CD'}$ and change its line width to dashed.

Q2 Why is $\overline{CD'}$ the shortest path from point C to point D'?

Q3 Where should point E be located in relation to $\overline{CD'}$ and $\overline{AB}$ so that the sum $CE + ED$ is minimized? Drag point E to test your conjecture.

Explore More

1. In a new sketch, use a reflection to construct a model of the burning tent problem so that the path from the hiker to the river to the tent is always minimal, no matter where you locate the hiker and the tent.
2. What's the shortest mirror you'd need on a wall in order to see your full reflection from your toes to the top of your head? To explore this question, construct a vertical segment representing you and another vertical segment representing a mirror. Construct a point to represent your eye level just below the top endpoint of the segment representing you. Use a reflection to construct the path a ray of light would take from the top of your head to the mirror and to your eye. Construct another path that a ray of light would take from your toes to the mirror to your eye. Adjust the mirror length so that it's just long enough for both light rays to reflect off it. How long is the mirror compared to your height?

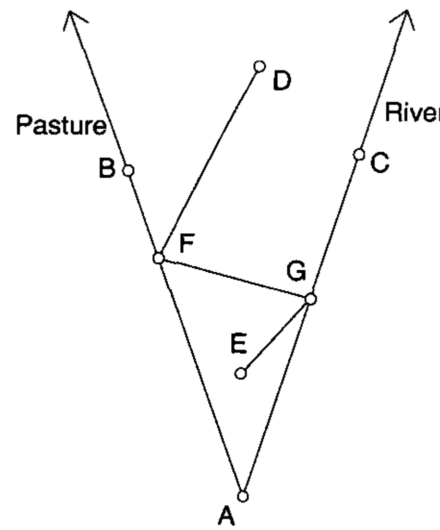


The Feed and Water Problem

Name(s): ____________________

A rider is traveling from point D to point E between a river and a pasture. Before he gets to E, he wants to stop at the pasture to feed his horse and at the river to water her. What path should he take? Use Sketchpad to model the problem.

Sketch and Investigate



1. Construct $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Using the **Text** tool, click once on the ray to show its label. Double-click the label to edit it.

2. Label these rays *Pasture* and *River*.
3. Construct points D and E between the rays.
4. Construct $\overline{DF}$ from point D to the pasture, $\overline{FG}$ from the pasture to the river, and $\overline{GE}$ from the river to point E.
5. Measure DF, FG, and GE.

Choose **Preferences** from the Edit menu and go to the Units panel to change precision settings.

6. Set Distance Precision to **thousandths**.

Choose **Calculate** from the Measure menu to open the Calculator. Click once on a measurement in your sketch to enter it into a calculation.

7. Calculate the sum $DF + FG + GE$.
8. Drag points F and G to minimize this sum. (You may have to drag each point several times.)
9. Measure angles DFB, AFG, CGF, and EGA.

Q1 What do you notice about the incoming and outgoing angles the path makes with the pasture and the river when the path is minimized? See if you can use the angle measures, as you drag points F and G, to make the path even shorter.

Explore More

1. Construct your solution so that it works wherever points D and E are located. In a new sketch, construct the rays and points D and E between them. Reflect point D across the pasture. Reflect point E across the river. The distance $D'E'$ is the same as the shortest distance from D to E via the pasture and the river. (Why?) Construct $\overline{D'E'}$ and use it to construct the shortest path from D to E via the pasture and the river. Hide D', E', and $\overline{D'E'}$. You should be able to move D and E and have the path change automatically to minimize the distance.

Planning a Path for a Laser

Name(s): ____________________

In this activity, you will use Sketchpad to model the path of a laser beam reflecting off three mirrors and striking a target. If you have a laser and some mirrors, you can test your model on the walls of your classroom.

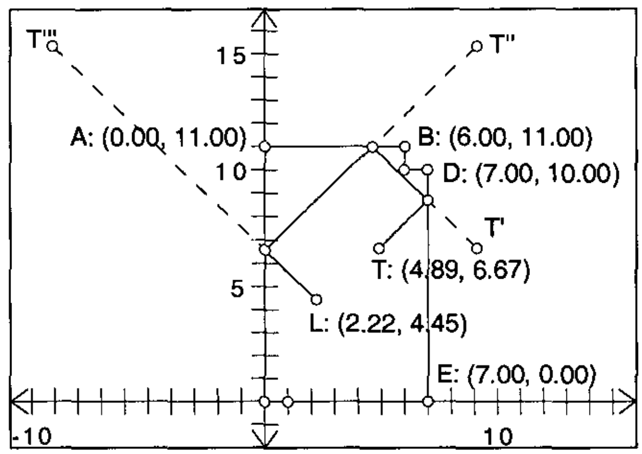
Sketch and Investigate

1. Make careful measurements of the dimensions of your classroom.

2. Choose one corner of the room to be the origin. This point will have the coordinates (0, 0). Figure out the coordinates of all the other vertices of your room. It will be most convenient to make your measurements in meters.

In the Graph menu, choose **Plot Points**. Enter the coordinates in the Plot Points dialog box and click **Plot**. Click **Done** when you're finished.

3. In Sketchpad, plot the points representing the vertices of your room. You may need to drag the point (1, 0) to scale your axes so you can see all the points.



4. Draw segments in your sketch to represent the walls of your room.

5. The laser beam should reflect off three walls of the room and strike a target. Use reflections to construct this path in your sketch. For now, your laser and target point can be anywhere inside the room. You will adjust their positions later.

6. Drag the laser and target points until they are positioned the way you want them. Now figure out where you would locate the mirrors, the actual laser, and the actual target if you were to test your model in your classroom.

Q1 Explain your construction. How did you find the locations for the mirrors?

Explore More

1. Find a laser and some mirrors and give your instructions to another group of students to test your plan. Make sure that the room is dark and that the laser beam is not at anyone's eye level. Report on the success of your plan.

Reflections over Two Parallel Lines

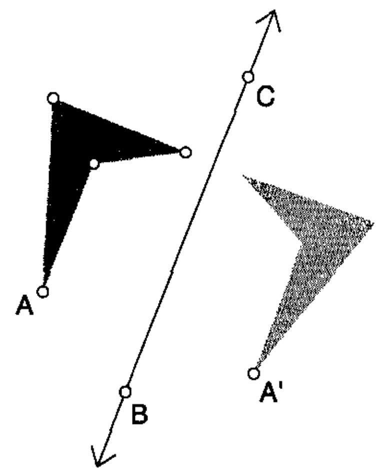
Name(s): ____________________

In this investigation, you'll see what happens when you reflect a figure over a line then reflect the image over a second line parallel to the first.

Sketch and Investigate

Hold down the Shift key while you click with the **Point** tool so that points will stay selected as you construct them. Then, in the Construct menu, choose **Quadrilateral Interior**.

1. Construct any irregular polygon interior.



With the **Text** tool, click on a point to display its label. Double-click a label to change it.

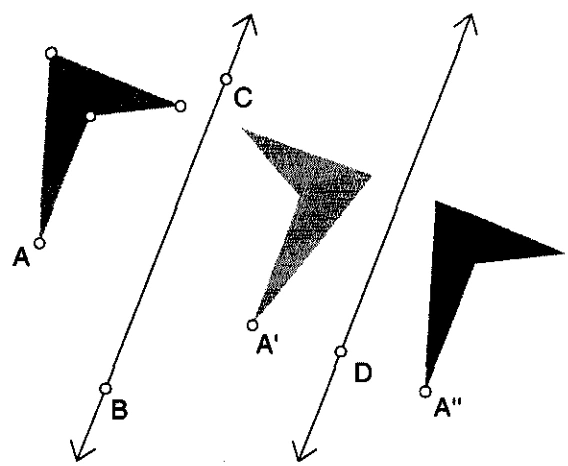
2. Show the label of one of the polygon's vertices. Change the label to *A* (if necessary).
3. Construct a line *BC*.

Double-click the line to mark it as a mirror. Select the polygon interior and the point; then, in the Transform menu, choose **Reflect**.

4. Mark the line as a mirror and reflect the polygon and point *A* over it.
5. Construct point *D* and a line through point *D* parallel to $\overleftrightarrow{BC}$.

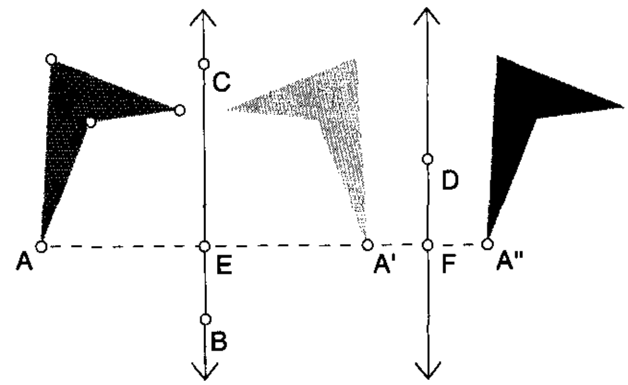
Select point *D* and $\overleftrightarrow{BC}$; then, in the Construct menu, choose **Parallel Line**.

6. Mark this second line as a mirror and reflect the first reflected image and point *A*′ over it.
7. Drag the original figure and the two lines and observe their relationships to the two images.



Q1 Two reflections move your original figure to its second image. What single transformation do you think will do the same thing? (If you're not sure, go on to the next steps, then come back to this question.)

8. Construct $\overline{AA''}$.



9. Construct points *E* and *F* where $\overline{AA''}$ intersects the two lines.

Select points *A* and *A''*; then, in the Measure menu, choose **Distance**. Repeat for *EF*.

10. Measure *AA''* and *EF*.

Q2 Drag one of the lines and compare the two distances. How are they related?

Q3 *EF* is the distance between the two lines. Why?

Select points *A* and *A''* in order; then, in the Transform menu, choose **Mark Vector**. Select the figure; then, in the Transform menu, choose **Translate**.

11. Mark *AA''* as a vector, then translate the original figure by this vector.

Q4 Describe the result of step 11, above. What single transformation is equivalent to the combination of two reflections over parallel lines?

Q5 Answer the following questions to explain why *AA''* and *EF* are related as they are:

a. How does *AE* compare to *EA'*? ____________________

b. How does *A'F* compare to *FA''*? ____________________

c. $AA' + A'A'' =$ ______________

d. Complete the rest of the explanation on your own.

Explore More

1. In the same sketch, try reflecting your figure and then its image over the two lines in the opposite order. Describe the result.

Reflections over Two Intersecting Lines

Name(s): ____________________

Hold down the Shift key while you click with the **Point** tool so that points will stay selected as you construct them. Then, in the Construct menu, choose **Quadrilateral Interior**.

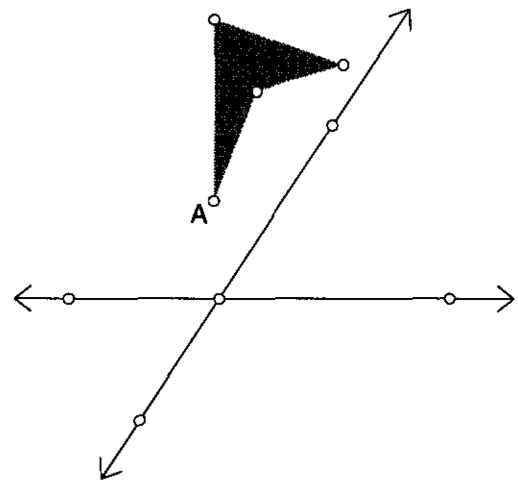
With the **Text** tool, click on a point to display its label. Double-click a label to change it.

Double-click the line to mark it as a mirror. Select the polygon and the point; then, in the Transform menu, choose **Reflect**.

In this investigation, you'll see what happens when you reflect a figure over a line and then reflect the image over a second line that intersects the first.

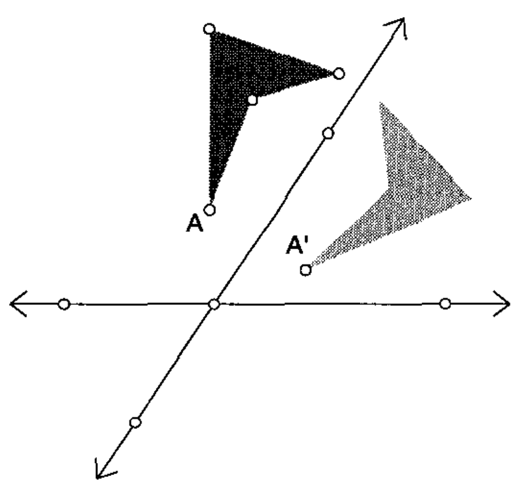
Sketch and Investigate

1. Construct any irregular polygon interior.
2. Show the label of one of the polygon's vertices and change it to A (if necessary).
3. Construct two intersecting lines and their point of intersection.

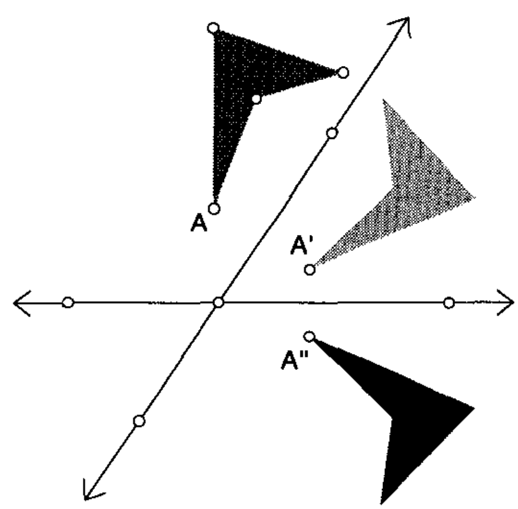


Steps 1–3

4. Mark the line closest to the polygon as a mirror, then reflect the polygon and the labeled point over this line. Change the color of the reflected image. (See the figure below left. If necessary, move the polygon so that the image falls between the lines.)
5. Mark the other line as a mirror, then reflect the image from the first reflection over this second line. Change the color of this second image. (See the figure below right.)



Step 4

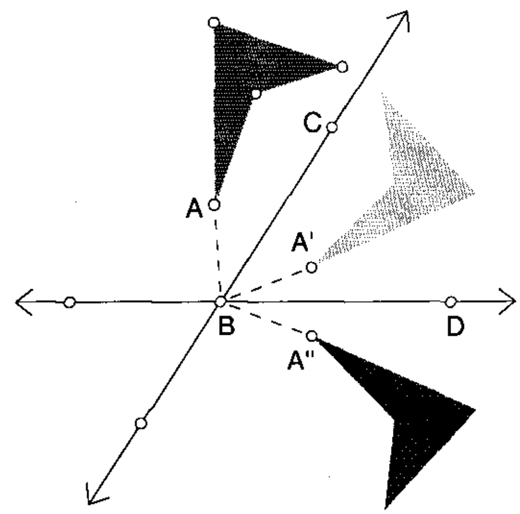


Step 5

6. Drag the original figure and the two lines and observe their relationships to the two images.

Q1 Two reflections move your original figure to its second image. What single transformation do you think will do the same thing? (If you're not sure, go on to the next steps, then come back to this question.)

7. Construct $\overline{AB}$, where point A is a point on the original figure and point B is the point of intersection of the lines.



8. Construct $\overline{BA'}$ and $\overline{BA''}$.

Select the three points that name the angle, with the vertex your middle selection; then, in the Measure menu, choose **Angle**.

9. Measure $\angle ABA''$.

10. Measure $\angle CBD$, the angle between the lines.

Q2 Compare the two angle measures. How are they related?

Double-click point *B* to mark it as a center. Select, in order, points *A*, *B*, and *A''*; then, in the Transform menu, choose **Mark Angle**. Select the figure; then, in the Transform menu, choose **Rotate**.

11. Mark point B as a center for rotation and mark $\angle ABA''$, then rotate the original figure by this angle.

Q3 Describe the result of step 11. What single transformation is equivalent to the combination of two reflections over intersecting lines?

Q4 Answer the following questions to explain why $m\angle ABA''$ and $m\angle CBD$ are related the way they are:

a. How does $m\angle ABC$ compare to $m\angle ABA'$? ____________________
(Try to answer without measuring first.)

b. How does $m\angle A'BD$ compare to $m\angle A'BA''$? ____________________

c. $m\angle ABA' + m\angle A'BA'' = m\angle$ __________

d. Complete the rest of the explanation on your own. Use a separate sheet, if necessary.

Glide Reflections

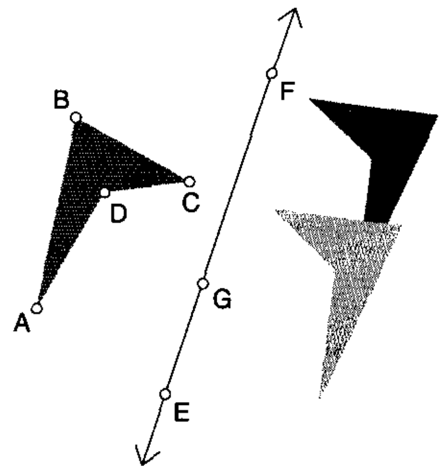
Name(s): ____________________

In this activity you will investigate an isometry called a *glide reflection*. Glide reflection is not a transformation found in the Transform menu, but you'll define it as a custom tool, and in the process you'll learn what a glide reflection is and what it does.

Sketch and Investigate

> Hold down the Shift key while you click with the **Point** tool so that points will stay selected as you construct them. Then, in the Construct menu, choose **Quadrilateral Interior**.

1. Construct an irregular polygon interior, like polygon *ABCD*, shown at right.



2. Construct line *EF*.

> Double-click the line to mark it as a mirror. Select the polygon interior; then, in the Transform menu, choose **Reflect**.

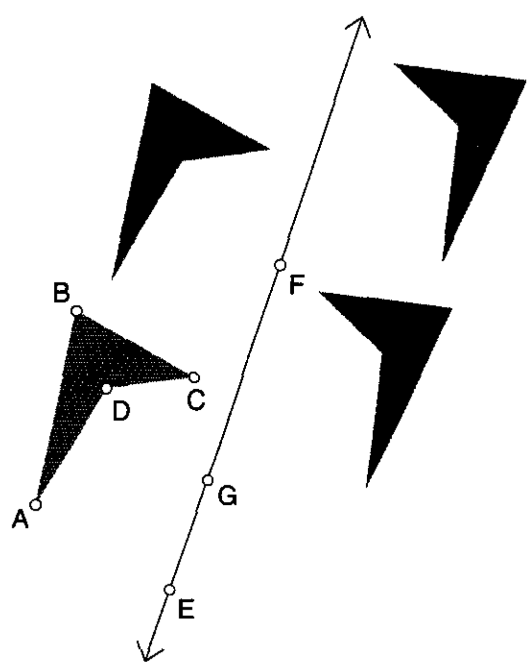
3. Mark $\overleftrightarrow{EF}$ as a mirror and reflect the polygon interior across $\overleftrightarrow{EF}$.
4. Construct a point *G* on the line so that *E* and *G* are about an inch apart.

> Select, in order, points *E* and *G*; then, in the Transform menu, choose **Mark Vector**. A brief animation indicates the mark. Select the reflected polygon interior; then, in the Transform menu, choose **Translate**.

5. Mark *EG* as a vector and translate the reflected image by the marked vector. This second image is a glide reflection of your original figure.
6. Drag point *G* to see how it affects the glide-reflected image.
7. Hide the intermediate image (the first reflection).

> Line *EF* and vector *EG* will remain marked; you don't have to mark them over and over again.

8. Use the techniques you've learned so far to create two more glide-reflected images, as shown at right.



9. Drag parts of your sketch (vertices of the original polygon, point *G*, points *E* and *F*, the line) and observe the effects.

Q1 A glide reflection is the product of two transformations. What are they?

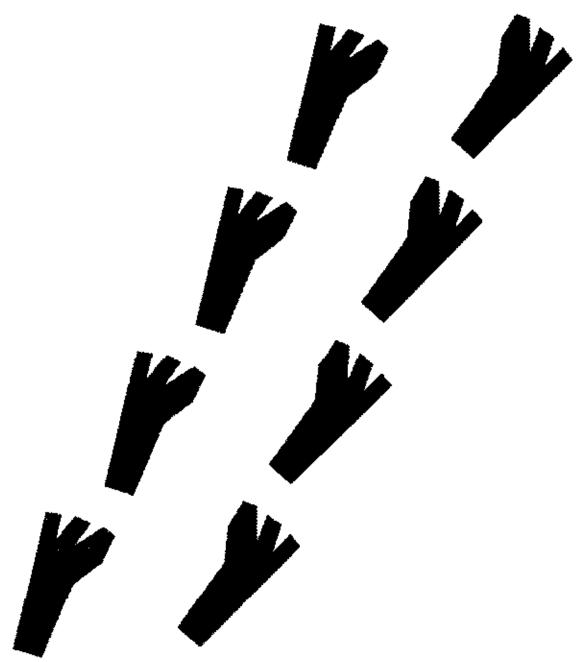
Q2 A translation is also the product of two transformations. What are they?

Q3 A glide reflection can be thought of as a product of what three transformations?

Explore More

For tips on making and using custom tools, choose **Toolbox** from the Help menu, then click on the Custom Tools link.

1. Turn your glide reflection into a custom tool to save yourself time when doing glide reflections. Save the tool in the Tool Folder (next to the application itself on your hard drive) for future use.
2. Create a polygon that looks like a foot, then use glide reflections to make a sketch that looks like footprints in sand. To make the footprints appear sequentially, as if made by a walking invisible person, follow these steps:



   a. Make a Hide/Show action button for each footprint.
   b. Make a Presentation button with the Hide/Show buttons. Choose Sequentially in the Presentation dialog box that appears, and add a one-second Pause Between Actions.
   c. Hide all of the footprints, then press the Presentation button.
3. Experiment with reflections across three random lines. Does this produce a glide reflection?
4. What's the product of a rotation and a reflection? What's the product of a rotation and a translation?

Symmetry in Regular Polygons

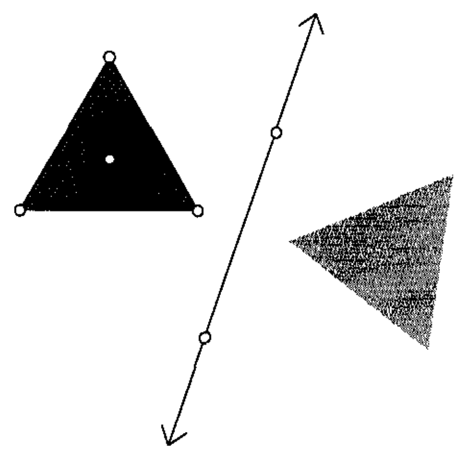
Name(s): ________________

A figure has *reflection symmetry* if you can reflect the figure over a line so that the image will coincide with the original figure. The line you reflect over is called a *line of symmetry* or a *mirror line*. A figure has *rotational symmetry* if you can rotate it some number of degrees about some point so that the rotated image will coincide with the original figure. In this exploration, you'll look for reflection and rotation symmetries of regular polygons.

Sketch and Investigate

Construct your polygon from scratch or use a custom tool. The sketch **Polygons.gsp** includes tools for regular polygons.

1. Construct a regular polygon and its interior. You can use an equilateral triangle, a square, a regular pentagon, or a regular hexagon. You may want to have different groups in your class investigate different shapes.



2. Construct a line.

Double-click the line to mark it as a mirror. Select the polygon interior; then, in the Transform menu, choose **Reflect**.

3. Mark the line as a mirror and reflect the polygon interior over it.
4. Give the image a different color.
5. Drag the line until the image of your polygon coincides exactly with the original.

Q1 When a reflection image coincides with the original figure, the reflection line is a line of symmetry. Describe how the line of symmetry is positioned relative to the figure.

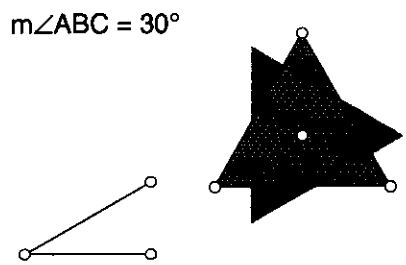
6. Drag the line so that it is a different line of symmetry. Repeat until you have found all the reflection symmetries of your polygon.

Q2 Fill in one entry in the table below: the number of reflection symmetries for your polygon. (*Note:* Be careful not to count the same line twice!) You'll come back to fill in other entries as you gather more information.

Number of sides of regular polygon	3	4	5	6	. . .	*n*
Number of reflection symmetries					. . .	
Number of rotation symmetries					. . .	

Next, you'll look for rotation symmetries.

7. Move the line so that the reflected image is out of the way.



8. If the polygon's center doesn't already exist, construct it.

9. Use the **Segment** tool to construct an angle.

10. Measure the angle.

11. In Preferences, set Angle Units to **directed degrees**.

12. Mark the center of the polygon as a center for rotation and mark the angle measurement. Rotate the polygon interior by this marked angle measurement.

13. Give the rotated image a different color.

14. Change the angle so that the rotated image fits exactly over the original figure.

Select the three points, with the vertex your middle selection. Then, in the Measure menu, choose **Angle**.

Choose **Preferences** from the Edit menu and go to the Units panel.

Double-click the point to mark it. Select the angle measurement; then, in the Transform menu, choose **Mark Angle**. Select the interior; then, in the Transform menu, choose **Rotate**.

Q3 What angle measure causes the figures to coincide?

Polygon: _______________ Rotation angle: _______________

15. Continue changing your angle to find all possible rotation symmetries of your polygon.

Q4 Count the number of times the rotated image coincides with the original when rotating from 0° to 180° and from –180° back to 0°. In your chart on the preceding page, record the total number of rotation symmetries you found. (*Note:* Count no revolutions, or 0°, as one of your rotation symmetries.)

Q5 Combine the results from other members of your class to complete your chart with the reflection and rotation symmetries of other regular polygons.

Q6 Use your findings to write a conjecture about the reflection and rotation symmetries of a regular *n*-gon. Include in your conjecture a statement about the smallest angle of rotational symmetry greater than 0.

Tessellating with Regular Polygons

Name(s): ____________

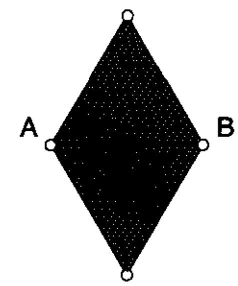
You've probably seen a floor tiled with square tiles. Squares make good tiles because they can cover a surface without any gaps or overlapping. This kind of tiling is sometimes called a *tessellation*. Are there other shapes that would make good tiles? In this investigation, you'll discover which regular polygons tessellate. You'll need custom tools for creating equilateral triangles, squares, regular pentagons, regular hexagons, and regular octagons. Each custom tool must create its figure from the endpoints of one side of the polygon—not from the center. Such custom tools come with the book, but you may want to create them yourself.

Sketch and Investigate

This step is unnecessary if you've previously placed this sketch in your Tool Folder.

1. Open the sketch **Polygons.gsp**.

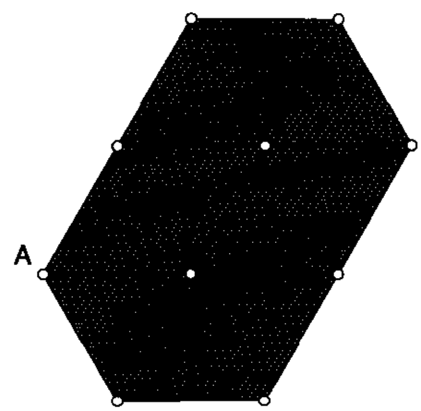
Click on the **Custom** tools icon (the bottom tool in the Toolbox) and choose the desired tool from the menu that appears (in the Polygons submenu). Click twice in the sketch to use the tool.



2. In a blank sketch, use a custom tool to construct an equilateral triangle. Pay attention to the direction in which the triangle is created as you use the tool.
3. Use the custom tool again, this time clicking in the opposite direction, to construct a second equilateral triangle attached to the first.

If your triangles don't stay attached, undo until the second triangle goes away, then try again. You must always start and end your dragging with the cursor positioned on an existing point.

4. Drag several points on the triangles to make sure they're really attached.
5. Keep attaching triangles to edges of existing triangles until you have triangles completely surrounding at least two points.



Q1 So far, you've demonstrated that equilateral triangles can tessellate. You can tile the plane with them without gaps or overlapping. Why do equilateral triangles work? (*Hint:* It has to do with their angles.)

6. Repeat the investigation with squares, regular pentagons, regular hexagons, and regular octagons.

Q2 Which of the regular polygons you tried will tessellate and which won't? Why? Answer on a separate sheet.

Q3 Do you think a regular heptagon (seven sides) would tessellate? Explain.

A Tumbling-Block Design

Name(s): ______________________

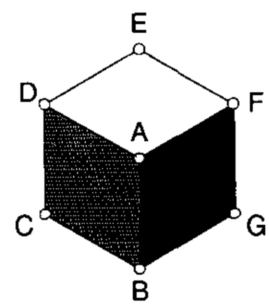
A tumbling-block design is commonly found in Amish quilt patterns. We can call it an example of "op art" because of its interesting optical effect, which is suggested by its name. A tumbling-block design can be created efficiently with Sketchpad using *translations*.

Sketch and Investigate

Use your own method or a custom tool like **6/Hexagon (Inscribed)** from the sketch **Polygons.gsp**.

1. Construct a regular hexagon.
2. Delete the polygon's interior, if necessary.
3. If the center of the hexagon doesn't already exist, construct it.
4. Construct segments and polygon interiors, and color regions of the hexagon as shown at right.

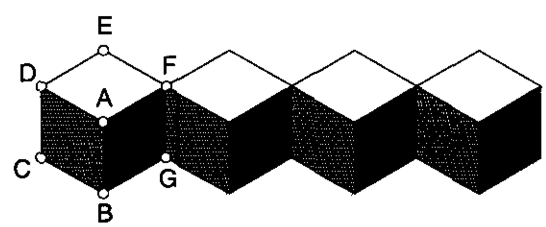
To construct a polygon interior, select the vertices in consecutive order; then, in the Construct menu, choose **Quadrilateral Interior**. **Color** is in the Display menu.



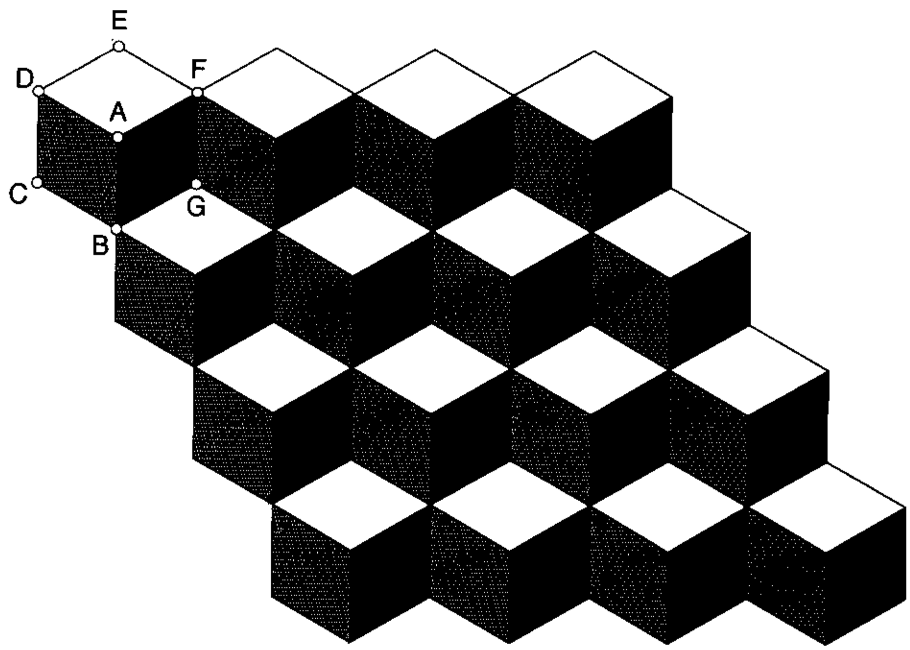
In the next few steps, you'll translate this figure to create a row of blocks.

Select, in order, point *D* and point *F*; then, in the Transform menu, choose **Mark Vector**.

5. Mark the vector *DF*.
6. Select the nine segments and two interiors (in other words, everything but the points).
7. Translate the selection.
8. Translate again two more times so that you have a row of four blocks.



9. Mark *EG* as a vector and translate the entire row of blocks (except the points) by this vector to create a second row of blocks.
10. Translate again two more times so that you have four rows of blocks.
11. Drag points on your original hexagon to scale and turn the design.
12. Experiment with different color patterns to enhance your design.



Q1 Describe different shapes in this design. What's the shape of the smallest pieces quilters use in this design?

Q2 Describe the optical effect of the design.

Q3 Describe any places where you have seen this design before.

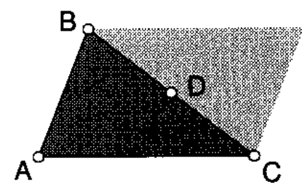
Tessellating with Triangles

Name(s): ____________________

In this investigation, you'll learn a method for tessellating with any triangle. You'll also discover why all triangles tessellate.

Sketch and Investigate

1. In the lower left corner of your screen, construct triangle *ABC*.



Select the three vertices; then, in the Construct menu, choose **Triangle Interior**.

2. Construct its interior.
3. Construct midpoint *D* of side *BC*.

Double-click the point to mark it as a center. Select the triangle interior; then, in the Transform menu, choose **Rotate**.

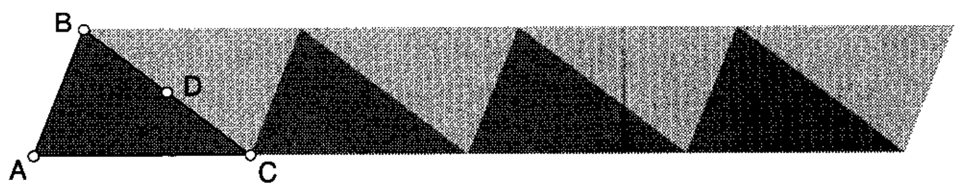
4. Mark point *D* as a center and rotate the triangle interior by 180°.
5. Give the rotated image a different color.

Color is in the Display menu.

Q1 Drag points and observe the shape formed by the two triangles (the original together with its rotated image). What shape is this? ____________________

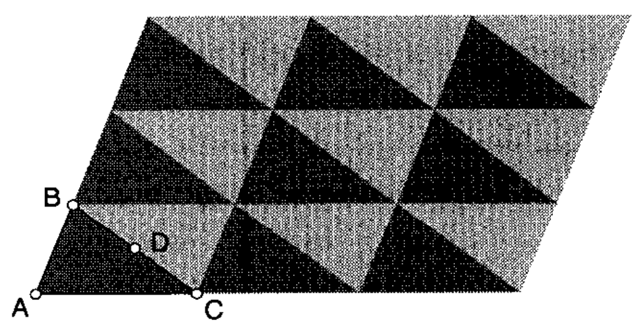
Select, in order, point *A* and point *C*; then, in the Transform menu, choose **Mark Vector**. A brief animation indicates the mark. Select the two interiors; then, in the Transform menu, choose **Translate**.

6. Mark *AC* as a vector and translate the two interiors by the marked vector. Translate two more times to make a total of eight triangles.



Q2 Drag to confirm that the top and bottom edges of this row of triangles are always straight lines. What does that demonstrate about the sum of the angle measures in the original triangle? Explain. (Write your explanation on a separate sheet.)

7. Mark vector *AB* and translate the entire row by this vector. Repeat until triangles begin to fill your screen.
8. Drag to confirm that no matter what shape your original triangle has, it will always tessellate.



Q3 Look at a point in the tessellation that is completely surrounded by triangles. What is the sum of the angles surrounding this point? Why? (Write your explanation on a separate sheet.)

Tessellations Using Only Translations

Name(s): ______________________

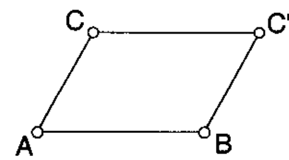
In this activity, you'll learn how to construct an irregularly shaped tile based on a parallelogram. Then you'll use translations to tessellate your screen with this tile.

Sketch

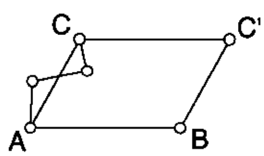
1. Construct $\overline{AB}$ in the lower-left corner of your sketch, then construct point C just above $\overline{AB}$.

Select, in order, point A and point B; then, in the Transform menu, choose **Mark Vector**. Select point C; then, in the Transform menu, choose **Translate**.

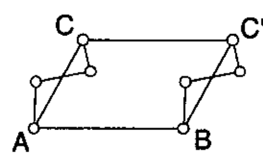
2. Mark the vector from point A to point B and translate point C by this vector.
3. Construct the remaining sides of your parallelogram.



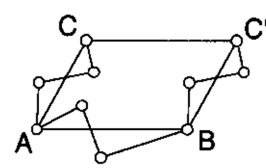
Steps 1–3



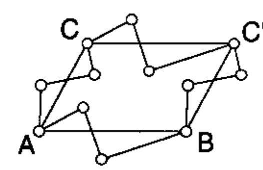
Step 4



Step 5



Step 6

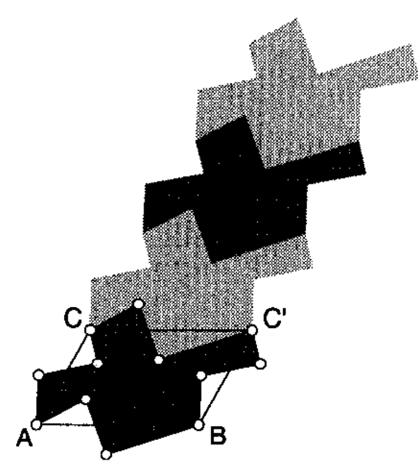


Step 7

4. Construct two or three connected segments from point A to point C. We'll call this *irregular edge AC*.
5. Select all the segments and points of irregular edge AC and translate them by the marked vector. (Vector AB should still be marked.)
6. Make an irregular edge from A to B.
7. Mark the vector from point A to point C and translate all the parts of irregular edge AB by the marked vector.

Select the vertices in consecutive order; then, in the Construct menu, choose **Polygon Interior**.

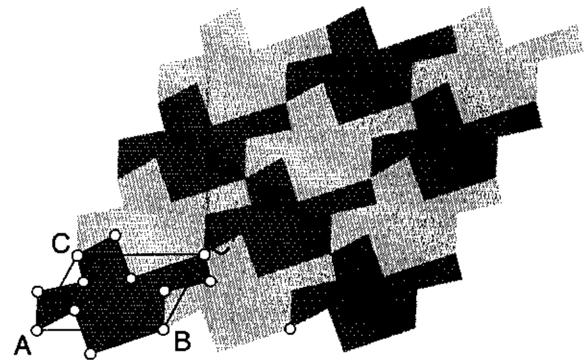
8. Construct the polygon interior of the irregular figure. This is the tile you will translate.
9. Translate the polygon interior by the marked vector. (You probably still have vector AC marked.)
10. Repeat this process until you have a column of tiles all the way up your sketch. Change the color of every other tile to create a pattern.



Steps 8–10

11. Mark vector AB. Then select all the polygon interiors in your column of tiles and translate them by this marked vector.

12. Continue translating columns of tiles until you fill your screen. Change colors of alternating tiles so you can see your tessellation.



Steps 11 and 12

13. Drag vertices of your original tile until you get a shape that you like or that is recognizable as some interesting form.

Explore More

1. Animate your tessellation. To do this, select the original polygon (or any combination of its vertex points) and choose **Animate** from the Display menu. You can also have your points move along paths you construct. To do this, construct the paths (segments, circles, polygon interiors—anything you can construct a point on) and then merge vertices to paths. (To merge a point to a path, select both and choose **Merge Point to Path** from the Edit menu.) Select the points you wish to animate and, in the Edit menu, choose **Action Buttons | Animation**. Press the Animation button. Adjust the paths so that the animation works in a way you like, then hide the paths.

2. Use Sketchpad to make a translation tessellation that starts with a regular hexagon as the basic shape instead of a parallelogram. (*Hint:* The process is very similar; it just involves a third pair of sides.)

Tessellations That Use Rotations

Name(s): ____________

Tessellations that use only translations have tiles that all face in the same direction. Using rotations, you can make a tessellation with tiles facing in different directions. The designs in a rotation tessellation have rotation symmetry about points in the tiling.

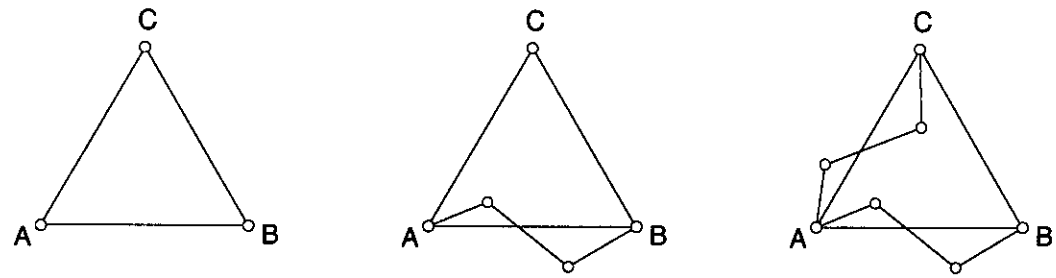
Sketch and Investigate

Use a custom tool (such as one in the sketch **Polygons.gsp**) or construct the triangle from scratch. If your custom tool constructs an interior, delete the interior.

1. Construct equilateral triangle *ABC* as shown below.
2. Construct two or three connected segments from *A* to *B*. We'll call this *irregular edge AB*.

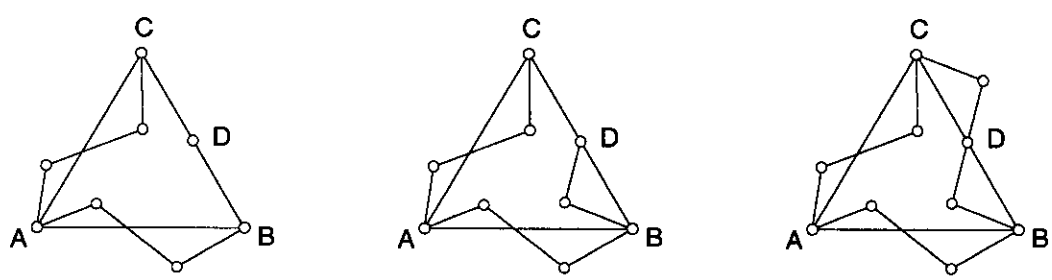
Double-click the point to mark it as a center. Select the segments and points; then, in the Transform menu, choose **Rotate**.

3. Mark point *A* as a center for rotation. Then rotate all the points and segments of irregular edge *AB* by 60°.



Step 1 Step 2 Step 3

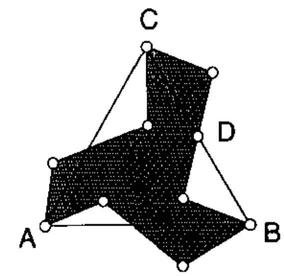
4. Construct midpoint *D* of side *CB*.
5. Construct two connected segments from *B* to *D*. We'll call this *irregular edge BD*.
6. Mark point *D* as a center for rotation. Then rotate the point and segments of irregular edge *BD* by 180°.
7. You have finished the edges of your tile. Drag points to see how they behave. When you're done, make sure none of the irregular edges intersect.



Step 4 Step 5 Step 6

Select the vertices in consecutive order; then, in the Construct menu, choose **Polygon Interior**.

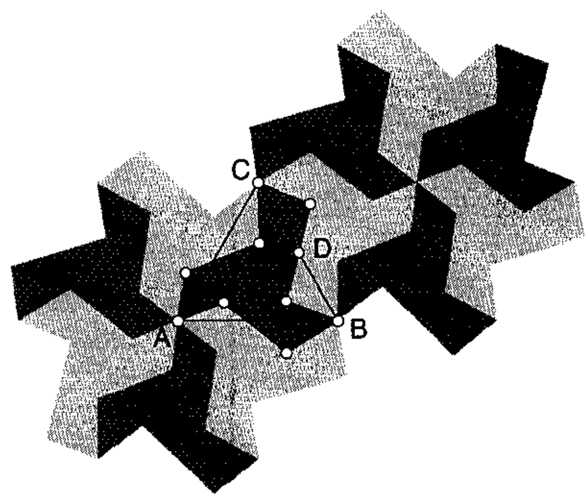
8. Construct the polygon interior with vertices along the irregular edges.



Change the color of your tiles using the Display menu.

9. To begin tessellating, mark point *A* as a center and rotate the tile interior six times by the appropriate number of degrees to surround point *A* with tiles. Change the color of alternate tiles.

10. Mark point *D* as a center and rotate the six tiles by 180°. Reverse their shading as necessary to keep a clear shading pattern.



Q1 Look at the tiles surrounding point *A*. What kind of rotation symmetry would the completed tessellation have about this point?

Q2 Look at the tiles surrounding point *D*. What kind of rotation symmetry would the completed tessellation have about this point?

Q3 Look at the tiles surrounding points *B* and *C* so far. What angle of rotation will you have to use to fill in tiles around these points?

11. Use the appropriate rotations to fill in tiles around points *B* and *C*. If you choose an angle that doesn't work right, undo and try a different angle. Change your answer to Q3, if necessary.

12. Drag vertices of your original tile until you get a shape that you like or that is recognizable as some interesting form.

- Move among groups posing questions, giving help if needed, and keeping students on task.
- Summarize students' findings in a whole-class discussion to bring closure to the lesson.

A Computer Lab

Teachers using Sketchpad often find that even if enough computers are available for students to work individually, it's still best to have students work in pairs. Students learn best when they communicate about what they're learning, and students working together can better stimulate ideas and help one another. If you do have students working at their own computers, encourage them to talk about what they're doing and to compare their findings with those of their neighbors—they *should* peek over one another's shoulders. The suggestions above for students working in small groups apply to students working in pairs as well.

Exploring Geometry and Your Geometry Text

The variety of ways Sketchpad can be used makes it an ideal tool for exploring geometry, no matter what text you're using. Use Sketchpad to demonstrate concepts or example problems presented in the text. Or have students use Sketchpad to explore problems given as exercises. If your text presents theorems and proves them, give your students an opportunity to explore the concepts with Sketchpad before moving on to the proof. Working out constructions in Sketchpad will deepen students' understanding of geometry concepts and will make proof more relevant than it would be otherwise.

Sketchpad is ideally suited for use with books that take a discovery approach to learning geometry. In Michael Serra's *Discovering Geometry*, for example, students working in small groups do investigations and discover concepts for themselves before they attempt proof. Many of these investigations call for compass-and-straightedge constructions, any of which can be done in Sketchpad. Many other investigations involving measurements, calculations, and transformations can also be done effectively and efficiently with Sketchpad.

We wish to stress, however, that we don't advocate abandoning all other teaching methods in favor of using the computer. Students need a variety of learning experiences, such as hands-on manipulatives, compass-and-straightedge constructions, drawing, paper-and-pencil work, and discussion. Students also need to apply geometry to real-life situations and see where it is used in art and architecture and where it can be found in nature. While Sketchpad can serve as a medium for many of these experiences, its potential will be reached only when students can apply what they learn with it to different situations.

Common Commands and Shortcuts

Below are some common Sketchpad actions used throughout this book. In time, these operations will become familiar, but at first you may want to keep this list by your side.

To undo or redo a recent action

Choose **Undo** from the Edit menu. You can undo as many steps as you want, all the way back to the state your sketch was in when last opened. To redo, choose **Redo** from the Edit menu.

To deselect everything

Click in any blank area of your sketch with the **Arrow** tool or press Esc until objects deselect. To deselect a single object while keeping all other objects selected, click on it with the **Arrow** tool.

To show or hide a label

Position the finger of the **Text** tool over the *object* and click. The hand will turn black when it's correctly positioned to show or hide a label.

To change a label

Position the finger of the **Text** tool over the *label* and double-click. The letter "A" will appear in the hand when it's correctly positioned.

To change an object's line width or color

Select the object and choose from the appropriate submenu in the Display menu.

To hide an object

Select the object and choose **Hide** from the Display menu.

To select an angle

Select three points, making the angle vertex your middle selection. Three points are the required selections to construct an angle bisector (Construct menu), to measure an angle (Measure menu), or to mark an angle of rotation (Transform menu).

To mark a center of rotation or dilation

Use the **Selection Arrow** tool to double-click the point. A brief animation indicates that you've made the mark.

To construct a segment's midpoint

Select the segment and choose **Midpoint** from the Construct menu.

To construct a parallel or perpendicular line

Select a straight object for the new line to be parallel/perpendicular to and a point for it to pass through. Then choose **Parallel Line** or **Perpendicular Line** from the Construct menu.

To construct a polygon interior

Select the vertices (points) of the polygon in consecutive order around the polygon. Then, in the Construct menu, choose **[Polygon] Interior.**

To reflect a point (or other object)

Double-click the mirror (any straight object) or select it and choose **Mark Mirror.** Then select the point (or other object) and choose **Reflect** from the Transform menu.

To trace an object

Select the object and choose **Trace** from the Display menu. Do the same thing to toggle tracing off. To erase traces left by traced objects, choose **Erase Traces** from the Display menu.

To use the Calculator

Choose **Calculate** from the Measure menu. To enter a measurement into a calculation, click on the measurement itself in the sketch.

Keyboard Shortcuts

Command	Mac	Windows
Undo	⌘+Z	Ctrl+Z
Redo	⌘+R	Ctrl+R
Select All	⌘+A	Ctrl+A
Properties	⌘+?	Alt+?
Hide Objects	⌘+H	Ctrl+H
Show/Hide Labels	⌘+K	Ctrl+K
Trace Objects	⌘+T	Ctrl+T
Erase Traces	⌘+B	Ctrl+B

Command	Mac	Windows
Animate/Pause	⌘+`	Alt+`
Increase Speed	⌘+]	Alt+]
Decrease Speed	⌘+[	Alt+[
Midpoint	⌘+M	Ctrl+M
Intersection	⌘+I	Ctrl+I
Segment	⌘+L	Ctrl+L
Polygon Interior	⌘+P	Ctrl+P
Calculate	⌘+=	Alt+=

Action	Mac	Windows
Scroll drag	Option+ drag	Alt+drag
Display Context menu	Control+ click	Right+ click
Navigate Toolbox	Shift+arrow keys	
Choose Arrow, deselect objects, stop animations, erase traces	Esc (escape key)	
Move selected objects 1 pixel	←, ↑, →, ↓ keys (hold down to move continuously)	