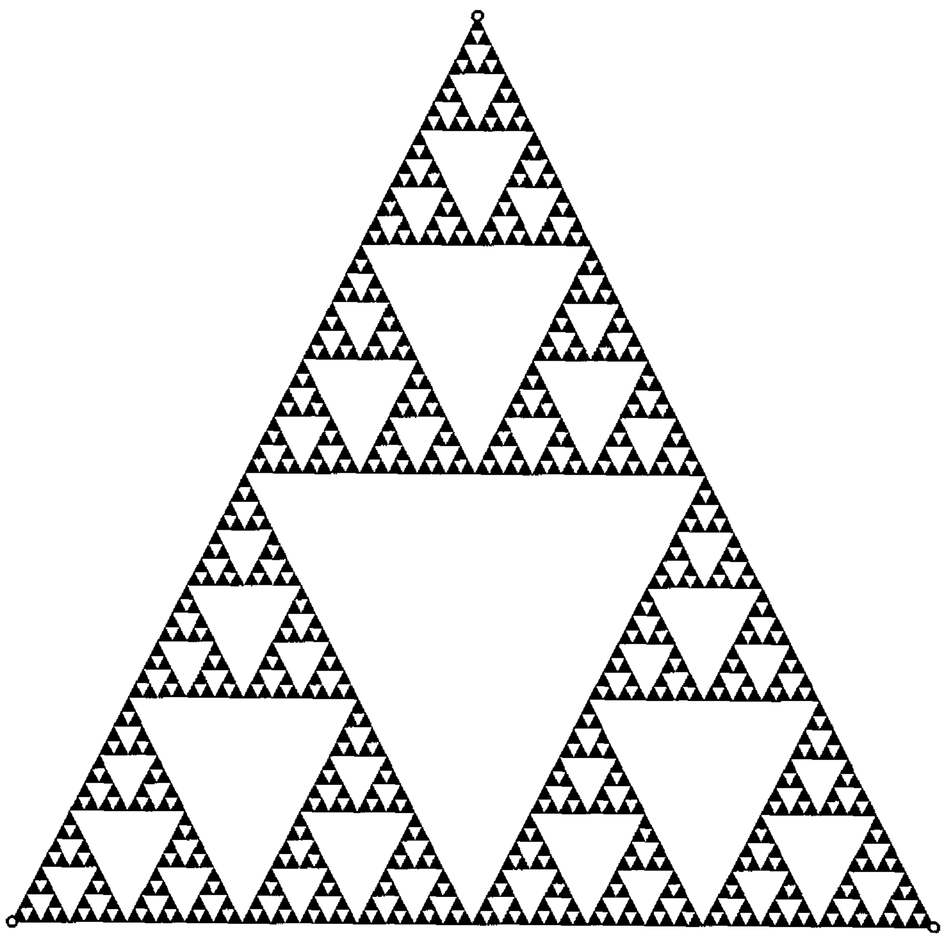
10

Trigonometry and Fractals



Trigonometric Ratios

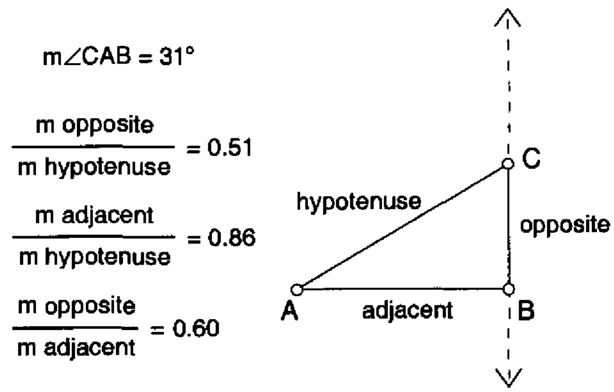
Name(s): ____________________

Right-triangle trigonometry builds on similar-triangle concepts to give you more ways to find unknown measures in triangles. In this activity, you'll learn about trigonometric ratios and how you can use them.

Sketch and Investigate

In steps 1–5, you'll construct a right triangle.

1. Construct $\overline{AB}$.



> Select point *B* and $\overline{AB}$; then, in the Construct menu, choose **Perpendicular Line**.

2. Construct a line through point *B* perpendicular to $\overline{AB}$.
3. Construct $\overline{AC}$, where point *C* is a point on the perpendicular line.
4. Hide the line.
5. Construct $\overline{BC}$ to finish the right triangle.

> Using the **Text** tool, click once on a segment to show its label. Double-click the label to change it.

6. Show the three segments' labels and change the labels to match the figure above right.

> Select, in order, points *C*, *A*, and *B*. Then, in the Measure menu, choose **Angle**.

7. Measure angle *CAB*.

> For each ratio, select the two segments in order. Then, in the Measure menu, choose **Ratio**.

8. Measure the ratios *opposite/hypotenuse, adjacent/hypotenuse,* and *opposite/adjacent.*

Q1 Drag point *C* to change the angles. When the angles change, do the ratios also change?

Q2 Drag point *A* or point *B* to scale the triangle. What do you notice about the ratios when the angles don't change? Explain why you think this happens.

Your observations in Q2 give you a useful fact about right triangles. For any right triangle with a given acute angle, each ratio of side lengths has a given value, regardless of the size of the triangle. The three ratios you measured are called *sine, cosine,* and *tangent.*

> Choose **Calculate** from the Measure menu to open the Calculator. In the Functions pop-up menu, choose **sin**. Click in the sketch on the measure of ∠*CAB*, then click OK. Use the same process to calculate cosine and tangent.

9. The sine, cosine, and tangent functions can be found on all scientific calculators, commonly abbreviated as sin, cos, and tan. Use Sketchpad's Calculator to calculate the sine, cosine, and tangent of ∠*CAB*. Match these calculations with the ratios they are equal to.

Q3 Complete the ratios for cosine and tangent below.

$$\text{sine } \angle A = \frac{\text{length of leg opposite } \angle A}{\text{length of hypotenuse}}$$

cosine $\angle A$ = ______________________

tangent $\angle A$ = ______________________

Q4 Drag point C so that $\angle A$ measures as close to 30° as you can get it. Write approximate values for the sine, cosine, and tangent of 30° below. Use the definitions in Q3 and refer to the calculations in your sketch to find these values.

sin 30° = ________ cos 30° = ________ tan 30° = ________

Q5 Without measuring, figure out the measure of $\angle C$ and write down that number. Calculate the sine of that angle measure. The sine of $\angle C$ should be close to one of the trigonometric ratios for $\angle A$. Which one? Explain why this is so.

Q6 Drag point C and answer the following questions.

a. What's the smallest possible value for the sine of an angle in a right triangle? What angle has this value? ____________

Hint: Make $\overline{AB}$ short so that you can drag point C up farther.

b. What's the greatest possible value for the sine of an angle in a right triangle? What angle has this value? ____________

c. Why can't you make m$\angle CAB$ exactly equal to 90°?

d. Even though you can't make m$\angle CAB$ exactly equal to 90°, what do you think is the value of tan 90°? Explain.

e. For what angle is the tangent equal to 1? Why?

f. For what angle are the sine and cosine equal? Why?

g. Suppose an angle has measure x. Complete this equation:

$\sin x = \cos$ ____________.

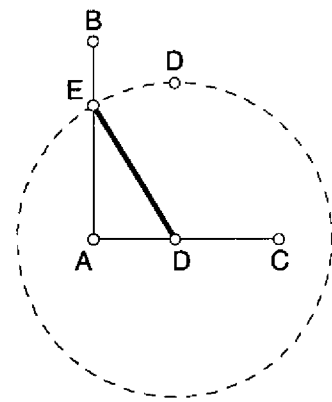
Modeling a Ladder Problem

Name(s): ____________________

Drawing diagrams is a useful method to help solve many types of realistic problems. Dynamic diagrams can be even more useful. Here's a problem that can be solved with a Sketchpad sketch.

The Occupational Safety and Health Administration (OSHA) recommends that when you use a ladder, you should lean it against a wall so that the height at which it touches the wall is four times the distance from the wall to the foot of the ladder. Any more and you risk tipping the ladder backward. Any less and you risk having the bottom slide out from under the ladder. What's the height from the floor that you can reach with a 20-foot ladder? What angle will the ladder make with the floor?

Sketch and Investigate



Choose **Preferences** from the Edit menu and go to the Units panel.

1. Set Preferences to display the Distance Units in **inches**.

Holding down the Shift key while you draw makes it easier to draw vertical and horizontal segments.

2. Construct vertical segment AB and horizontal segment AC. These segments represent the wall and the floor.
3. Construct point D on the floor. This point will be the foot of your ladder.

Select point *D*; then, in the Transform menu, choose **Translate**.

4. Translate point D vertically by 2 inches. The 2 inches will represent the length of your ladder, so the scale of your drawing will be 1 in. = 10 ft.
5. Construct circle DD'.
6. Construct point E where the circle intersects the wall. You may have to move point D first so that the circle and the wall intersect.
7. Construct $\overline{DE}$. This segment represents your ladder. Its length can't change because the radius of the circle is fixed at 2 inches.
8. Hide the circle and point D'.
9. Drag point D back and forth. You should see the top of the ladder move up and down the wall.

Select, in order, points *E*, *D*, and *B*. Then, in the Measure menu, choose **Angle**. Select points *E* and *A*; then, in the Measure menu, choose **Distance**. Repeat for *AD*.

10. Measure $\angle EDA$, EA, and AD. (EA represents the height on the wall that your ladder is reaching.) Calculate EA/AD.

Q1 Drag point D. Given the constraints in the problem, how high can the ladder reach? What angle does it make with the floor?

Q2 Confirm your answers using trigonometry. Show your work.

Explore More

1. Suppose a ladder is propped against one wall in the corner of a room. To one side of the ladder is another wall. A wet paintbrush rests on the center rung of the ladder, just touching the side wall. Suddenly, the foot of the ladder slips and the paintbrush falls with it, painting a streak on the side wall as it falls! What does the streak look like? To model this in your sketch, construct the midpoint of your ladder. While it's selected, choose **Trace Point** in the Display menu. Animate point *D* along $\overline{BC}$.
2. Select the measurements for *EA* and *AD* and choose **Plot As (x, y)** in the Graph menu. Drag the foot of the ladder. What kind of graph do you get? If you were to drag the foot of a ladder away from a wall at a constant rate, would the top of the ladder fall at a constant rate? Why or why not?
3. Write one or more other problems that could be modeled with this sketch.

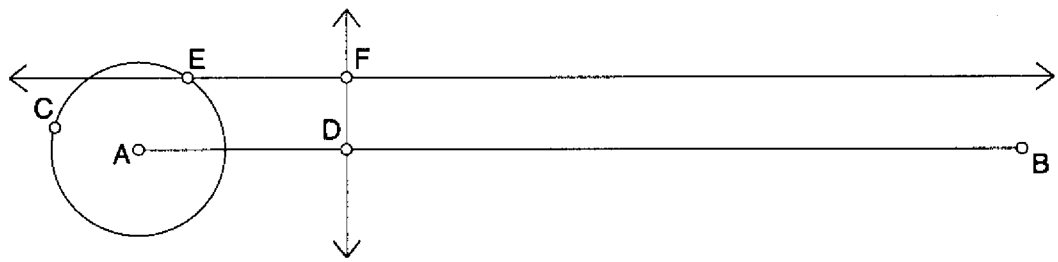
A Sine Wave Tracer

Name(s): ___________________________

In this exploration, you'll construct an animation "engine" that traces out a special curve called a *sine wave*. Variations of sine curves are the graphs of functions called *periodic functions*, functions that repeat themselves. The motion of a pendulum and ocean tides are examples of periodic functions.

Sketch and Investigate

1. Construct a horizontal segment AB.



2. Construct a circle with center A and radius endpoint C.
3. Construct point D on $\overline{AB}$.
4. Construct a line perpendicular to $\overline{AB}$ through point D.

   Select point D and $\overline{AB}$; then, in the Construct menu, choose **Perpendicular Line**.

5. Construct point E on the circle.
6. Construct a line parallel to $\overline{AB}$ through point E.
7. Construct point F, the point of intersection of the vertical line through point D and the horizontal line through point E.

Q1 Drag point D and describe what happens to point F.

Don't worry, this isn't a trick question!

Q2 Drag point E around the circle and describe what point F does.

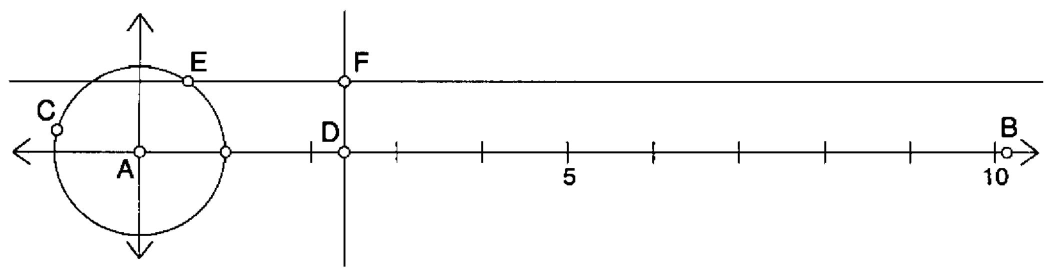
Q3 In a minute, you'll create an animation in your sketch that combines these two motions. But first try to guess what the path of point F will be when point D moves to the right along the segment at the same time that point E is moving around the circle. Sketch the path you imagine below.

Select points D and E and choose **Edit | Action Buttons | Animation**. Choose **forward** in the Direction pop-up menu for point D.

8. Make an action button that animates point D forward along $\overline{AB}$ and point E forward around the circle.
9. Move point D so that it's just to the right of the circle.
10. Select point F; then, in the Display menu, choose **Trace Point**.
11. Press the Animation button.

Q4 In the space below, sketch the path traced by point F. Does the actual path resemble your guess in Q3? How is it different?

12. Select the circle; then, in the Graph menu, choose **Define Unit Circle**. You should get a graph with the origin at point A. Point B should lie on the x-axis. The y-coordinate of point F above $\overline{AB}$ is the value of the sine of $\angle EAD$.



Q5 If the circle has a radius of 1 grid unit, what is its circumference in grid units? (Calculate this yourself; don't use Sketchpad to measure it because Sketchpad will measure in inches or centimeters, not grid units.)

13. Measure the coordinates of point B.
14. Adjust the segment and the circle until you can make the curve trace back on itself instead of drawing a new curve every time. (Keep point B on the x-axis.)

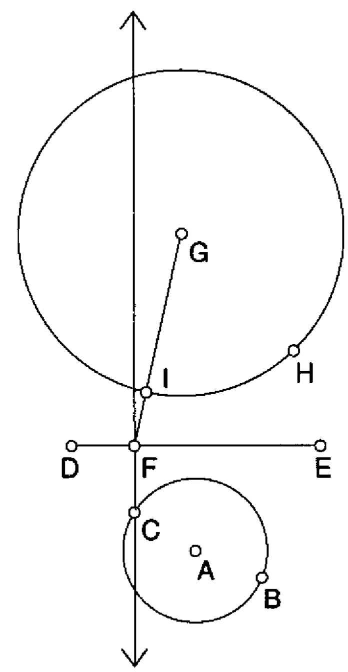
Q6 What's the relationship between the x-coordinate of point B and the circumference of the circle (in grid units)? Explain why you think this is so.

Modeling Pendulum Motion

Name(s): ____________________

A pendulum swings back and forth, slowing down to a momentary stop at each end of its swing and reaching its greatest speed at the bottom of its swing. This motion is periodic, and as long as the pendulum doesn't swing too far, the motion can be described with a sine function. If you did the activity A Sine Wave Tracer, the method for constructing a pendulum model will seem familiar to you.

Sketch and Investigate



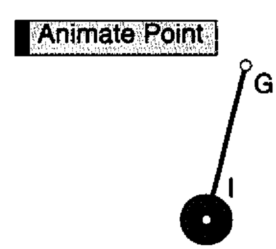
1. Construct a circle AB near the bottom of your sketch.
2. Construct point C on the circle.
3. Construct a horizontal segment DE above the circle.

Select point C and $\overline{DE}$; then, in the Construct menu, choose **Perpendicular Line**.

4. Construct a line through point C perpendicular to $\overline{DE}$.
5. Construct point F where this line intersects $\overline{DE}$.
6. Drag point C and observe the motion of point F. You'll use this back-and-forth motion to drive a pendulum.
7. Construct a large circle GH above $\overline{DE}$.
8. Construct $\overline{GF}$.
9. Construct point I where $\overline{GF}$ intersects the circle.
10. Drag point C and observe the motion of point I. Point I will be the bottom point of your pendulum.

Select point C; then choose **Edit | Action Buttons | Animation**.

11. Make an action button to animate point C around the circle.



12. Hide everything but the Animation button, point G, and point I.
13. Construct $\overline{GI}$. This is your pendulum string.
14. To make a nice pendulum bob, translate point I by 0.5 cm or 0.2 inches. Construct circle II' and its interior. Hide point I'.

If you wish to adjust the pendulum's speed, select the action button; then, in the Edit menu, choose **Properties**. Go to the Animate panel and use the Speed pop-up menu.

15. Press the Animation button to set your pendulum in motion. Press it again to stop the motion.

Creating a Hat Curve Fractal

Name(s): ____________________

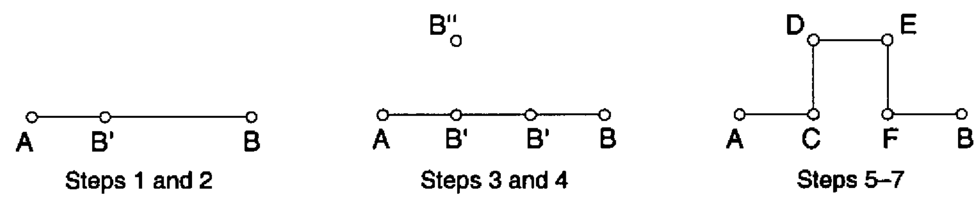
By using the **Iterate** command, you can repeat an action again and again on the same figure. Repeatedly taking the result of an action and applying the action to that result again is called *iteration* in mathematics and is central to the creation of fractals. In this activity, you'll create the first few stages of a fractal called the *hat curve.* A true hat curve, created in infinitely many stages, has dimension between 1 and 2.

Sketch and Investigate

1. In a new sketch, construct a long horizontal segment $\overline{AB}$ near the bottom of the sketch window.

Double-click point *A* to mark it as a center. Select point *B*; then, in the Transform menu, choose **Dilate**.

2. Mark point *A* as a center and dilate point *B* by a scale factor of 1/3.



3. Also dilate point *B* by a scale factor of 2/3.
4. Mark the first (left) point *B´* as a center and rotate the other (right) point *B´* by 90°.
5. Mark point *B´´* as a center and rotate the first (left) point *B´* by 90°.
6. Hide $\overline{AB}$.

Using the **Text** tool, click once on a point to show its label. Double-click the label to change it.

7. Connect the six points with segments to form a hat, as shown above right. Relabel the points to match the figure.

Take a moment to think about what you've "done" to segment *AB:* you've transformed it from a simple segment to a hat shape made up of five smaller segments. Now imagine doing the same thing (*iterating* the construction) to segment *AC,* and then to the other four smaller segments. Make a drawing in the margin of what the resulting figure (the *first iteration*) would look like.

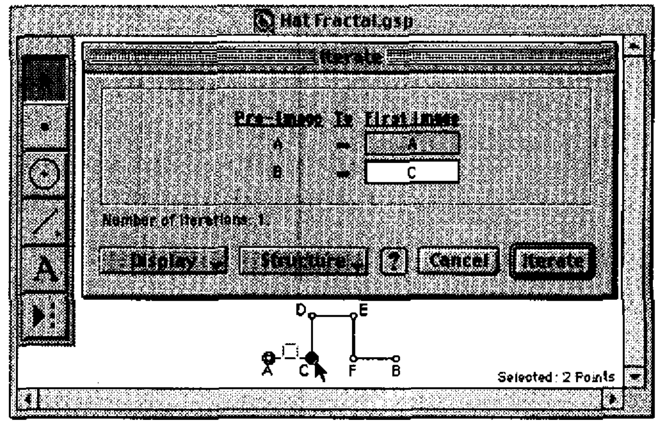
Can you imagine what the second iteration would look like (in other words, if you now applied the iteration rule to each of the very small segments in your drawing above)? You'll now use Sketchpad's **Iterate** command to construct an iterated image that can show several different stages of the iteration.

8. Select, in order, point *A* and point *B*. Choose **Iterate** from the Transform menu. The Iterate dialog box appears.
9. Choose **Final Iteration Only** from the Display pop-up menu. This tells Sketchpad to hide segments once they get iterated on.

Drag the dialog box by its title bar if you can't see a point in the sketch below.

10. Click first on point A in the sketch, then on point C. This tells Sketchpad to "do to segment AC what was done to segment AB."

11. Press the – key twice so that you see just one iteration.



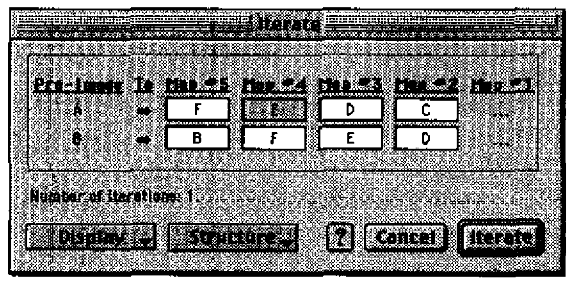
After step 11

12. Choose **Add New Map** from the Structure menu so that you can iterate on a new segment.

13. Click on point C first, then on point D.

If one of the cells in your dialog box has the wrong letter, highlight it and click on the proper point in the sketch.

14. Repeat steps 12 and 13 three more times, once for each remaining segment. When you're done, the Iterate dialog box should look like the one at right. Click Iterate.

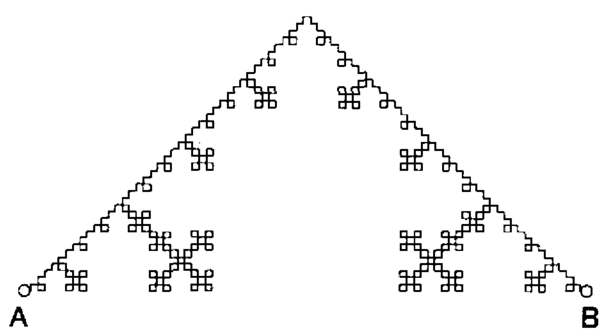


15. Hide segments AC, CD, DE, EF, and FB. You now should see a stage 1 hat curve. How does it compare to your drawing on the previous page?

16. Explore other stages of the hat curve by selecting part of the iterated image and pressing the + and – keys.

Q1 Sketch a stage 2 hat curve on a blank piece of paper.

Q2 What stage hat curve is shown at right?



Q3 Suppose the original segment AB had length 1 unit. A stage 0 hat curve is five segments, each 1/3 the length of the original, so its length is 5/3. Complete the chart below. What would be the length of a stage infinity hat curve?

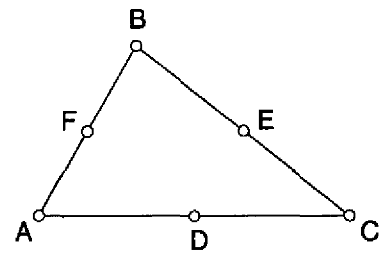
Stage	0	1	2	3	n
Length	5/3				

Creating a Sierpiński Gasket Fractal

Name(s): ______________

By using the **Iterate** command, you can repeat an action again and again on the same figure. Repeatedly taking the result of an action and applying the action to that result again is called *iteration* in mathematics and is central to the creation of fractals. In this activity, you'll create the first few stages of a fractal called the *Sierpiński gasket.* A true Sierpiński gasket, created in infinitely many stages, has dimension between 1 and 2 and its perimeter and area have strange properties.

Sketch and Investigate

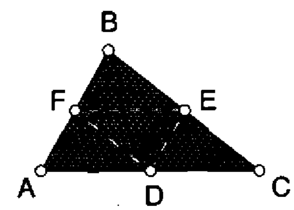


1. In a new sketch, construct a triangle ABC, with point A at the bottom left corner.
2. Construct the midpoints of the three sides.
3. If necessary, change labels to match the diagram.

Using the **Text** tool, click once on a point to show its label. Double-click the label to change it.

4. Construct the interior of the triangle.

Select the vertices; then, in the Construct menu, choose **Triangle Interior**.



As shown at right, the three midpoints help define four "subtriangles" of ΔABC, namely ΔAFD, ΔFBE, ΔDEC, and ΔEDF. (The dotted lines are shown for illustration only—they shouldn't be in your sketch.)

Take a moment to think about what you've "done" to triangle ABC: You've constructed its midpoints and its interior. Now imagine doing the same thing (*iterating* the construction) to ΔAFD, then to ΔFBE and ΔDEC (but *not* to ΔEDF), and then hiding the original interior. Make a drawing below of what the resulting figure (the *first iteration*) would look like.

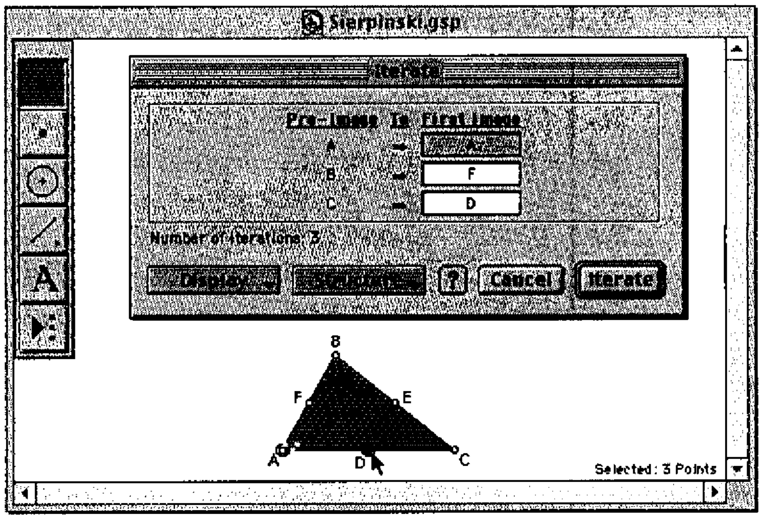
Can you imagine what the second iteration would look like (in other words, if you now applied the iteration rule to the three outer subtriangles in your drawing above)? You'll now use Sketchpad's **Iterate** command to construct an iterated image that can show several stages of the iteration.

5. Select, in order, points A, B, and C. Choose **Iterate** from the Transform menu. The Iterate dialog box appears.
6. Choose **Final Iteration Only** from the Display pop-up menu. This tells Sketchpad to hide triangle interiors once they get iterated on.

Creating a Sierpiński Gasket Fractal (continued)

Drag the dialog box by its title bar if you can't see a point in the sketch.

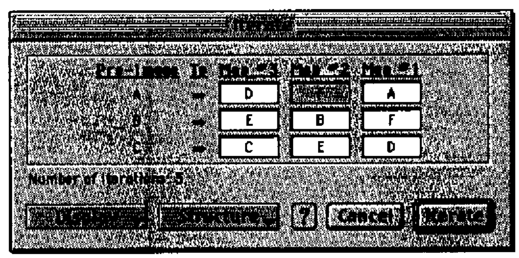
7. Click on point A first, then on point F, and then on point D. This tells Sketchpad to "do to triangle AFD what was done to triangle ABC."



After step 7

8. Choose **Add New Map** from the Structure menu.

9. Repeat step 7 for points F, B, and E, choose **Add New Map**, then repeat step 7 for points D, E, and C. When you're done, the Iterate dialog box should look like the one at right. Click Iterate.



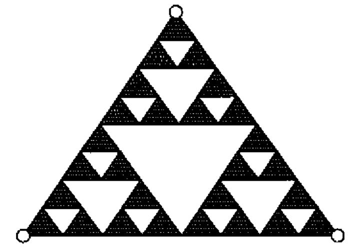
10. Deselect all objects. Now click in the center of ΔABC to select its interior, then hide it.

11. Hide the midpoints.

12. Select the iterated image and press the – key until nothing changes. This is the stage 1 iteration. How does it compare with your drawing on the previous page?

Q1 With the iterated image still selected, press the + key once to show the stage 2 iteration. Sketch the resulting figure below.

13. Use the + and – keys to explore other stages of the Sierpiński iteration.



Q2 What stage gasket is shown in the figure at right? ____________

Q3 Suppose a stage 0 gasket (a single triangle) has area 1 square unit. Fill in the middle row of the chart with the areas of stage 1, 2, 3, and 4 gaskets. As you increase the number of stages, what's happening to the area of the figure (the shaded part)? Enter an expression in the chart for the area of a stage *n* gasket.

Stage	0	1	2	3	4	*n*
Area	1					
Perimeter	3					

Q4 Suppose the stage 0 gasket started with perimeter 3 units. What happens to the perimeter, including the perimeters around all the little shaded triangles inside, as the stages increase? Complete the bottom row of the chart for perimeter.

Q5 What would the area of a Sierpiński gasket be at stage infinity? ____________

Q6 What would the perimeter of a Sierpiński gasket be at stage infinity? ____________