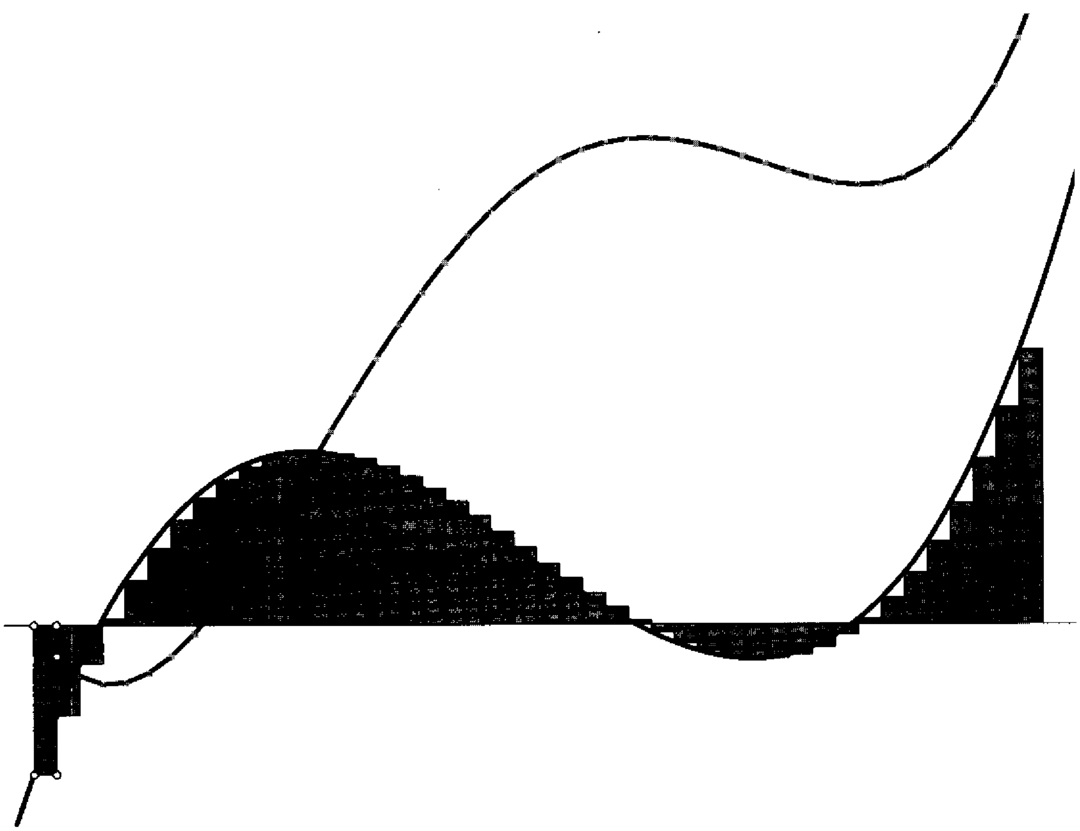
5

Exploring Integrals



Building Area

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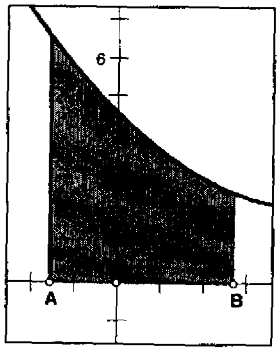


Finding the area between the plot of a function and the x-axis has a surprising number of applications. You have seen that the area under a velocity curve is the object's change in position, or the net distance traveled over the given time interval.

In this activity you will find the approximate area between a function plot and the x-axis using three different methods and tools. If you'd like to build these tools yourself, go to the Extension section of the Activity Notes after step 8.

Sketch and Investigate

1. **Open** the document **BuildArea.gsp** in the **Exploring Integrals** folder.

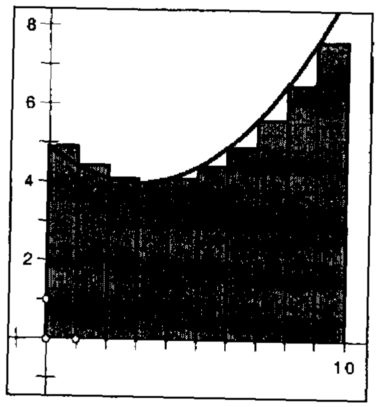
On page 1 of this document you will find the plot of a function $f(x)$, a slider that adjusts n, and an example of approximating the area between the plot of $f(x)$ and the x-axis from $x = 0$ to $x = 10$ by using rectangles.

Here n represents the number of rectangles.

2. Experiment with the slider that adjusts n. Make sure to try moving point N left and right of the segment that marks $n = 0$.

Q1 The rectangles are built differently, depending on whether point N is left or right of $n = 0$. Describe this difference.

Q2 Describe what happens to the rectangles, the area approximation, and the error as n increases.



Using rectangles may underestimate or overestimate the area, but the smaller the widths, the better the total sum of the rectangles' area will approximate the actual area. In the steps below, you will build the rectangles yourself and find the sum of their areas on your own.

If a, b, or c is not as given in step 3, press the *Set Quadratic* button.

3. Go to page 2 of the document. Here $f(x) = a(x - b)^2 + c$, where $a = 0.5$, $b = 3$, and $c = 2$.

4. To make an interval on the x-axis, choose the **Point** tool and construct two points on the x-axis at approximately $x = -0.5$ and $x = 5.5$. Label these points A and B.

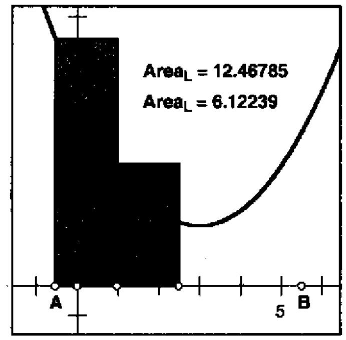
5. Select both points and then measure their x-coordinates by choosing **Measure | Abscissa (x)**.

Click on measurements to enter them into the calculator.

6. Choose **Measure | Calculate** and calculate the value $x_B - x_A$. This will be the width of the interval over which you will construct rectangles.

7. To divide the interval from point A to point B into n equal subdivisions, enter $(x_B - x_A)/n$ in the calculator. With the **Text** tool, label the result h. This will be the width of each of the n rectangles.

8. Choose **Left Rectangles** from **Custom** tools. Then click on point A.



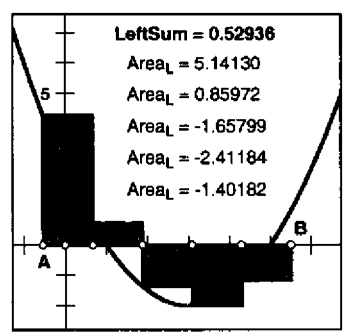
The tool will create a rectangle with the function's value on the left side of the subdivision as its height. The sum of the areas of a series of such rectangles is called a *left sum*.

9. To make the next rectangle, click on the new point on the x-axis (that was created in the last step) so that the two rectangles share the same base point, as shown at right.

10. Continue to make rectangles until you have covered the interval from point A to point B. When you are done, you should have n rectangles.

11. Find the sum of the n area calculations by choosing **Measure | Calculate** and then entering each $Area_L$ measurement. Label this measurement *LeftSum* and save this calculation.

12. The area calculated by the tool is the product of the width of the rectangle (h) and the height (the value of the function at the left endpoint of the subdivision). Adjust the slider for c to translate the function vertically. Watch the area calculations as you raise and lower the function, especially when the function takes on negative values.



Q3 Why are some of your $Area_L$ measurements now negative? What can you do to make the measurement *LeftSum* negative?

Q4 Given the equation of the function and a calculator, how would you produce the area measurements yourself?

Allowing negative areas (something you may never have considered before!) has both theoretical and practical uses, as you will see later on.

Next, you will create a *right sum*.

If needed, press the *Reset* button to reset the domain to [−0.5, 5.5].

13. Go to page 3 of the document. On this page you will find the same function that was on page 2, points A and B, and the calculation h. Points A and B are approximately in the same positions on the x-axis as they were originally on page 2.

14. Choose **Right Rectangles** from **Custom** tools. Then click on point A.

Q5 How is a right sum rectangle different from a left sum rectangle?

Here, the values for the sliders should still be $a = 0.5$, $b = 3$, and $c = 2$. If not, press the *Reset* button.

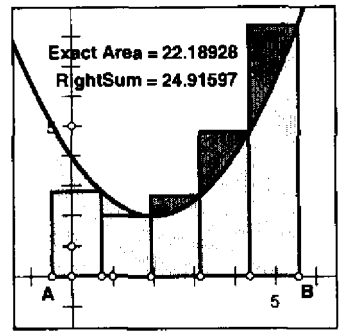
15. Continue to make rectangles as you did above in step 10. Make sure your value for n is the same as it was on page 2.

16. Find the sum of the n calculations of $Area_R$. Label it *RightSum* and save this calculation as well.

You wrote down your left sum in step 11.

Q6 How does the right sum compare with the left sum? Is one an underestimate and the other an overestimate?

17. Press the *Show Area* button to see both the exact region between the plot of $f(x)$ and the x-axis and its exact area.



Q7 How does the right sum compare with the exact area? How does the left sum compare?

Remember that $\%Error = (RS - exact)/exact$, where *RS* means *RightSum* and *exact* is the actual area.

Q8 Use the calculator to find the percentage error for the right sum and write your result in the margin.

You can modify the function by adjusting the sliders for a, b, or c, or changing the interval $[x_A, x_B]$ by dragging point A or point B. Observe how the right sum and the exact area vary for different functions or on different intervals. Can you find a function where the right sum is always an overestimate? An underestimate?

You can also increase the value of n, but you will have to add rectangles and recalculate the sum. As you saw in Q2, increasing the number of rectangles gives a more accurate estimate of the area.

To edit the *RightSum*, double-click on the calculation and add in the new $Area_R$ measurements.

18. Increase the number of subdivisions by adjusting the slider for n. Fill in the rest of the interval by adding the rectangles you need using the **Right Rectangles** tool. Find the new sum.

Q9 How much did your percentage error drop?

Explore More

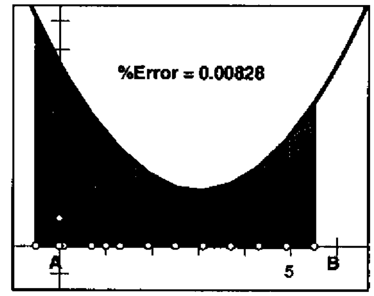
As you saw above, increasing the number of rectangles gives a more accurate estimate of the area. Another way to improve accuracy for your area estimate is by using trapezoids or midpoint sums. In the steps below, you will first build trapezoids, then another type of rectangle.

Check that this is indeed the function and domain. If not, press the *Reset* button.

1. Go to page 4 of the document. Here, you will be estimating the area under the curve $f(x) = 0.5(x - 3)^2 - 2$ on the interval $[-0.5, 5.5]$ again.

Make sure n is the same value you used back in steps 11 and 16.

2. Choose **Trapezoids** from **Custom** tools. Use this tool in the same way you used the other tools above to construct n trapezoids on the given interval and then calculate the approximate sum of the n trapezoids you constructed.



Q1 Looking at one trapezoid, what is the formula for the area using h and the function's values?

Q2 You can see graphically that the trapezoid sum gives a more accurate estimate of the area. Press the *Show Area* button and calculate the percentage error using trapezoids. Save this answer.

3. Press the Shift key, and then choose **Undo All** from the Edit menu.

4. Choose **Midpoint Rectangles** from **Custom** tools. Use this tool in the same way you used the other tools above to construct n rectangles on the given interval and then calculate the approximate sum of the n rectangles you constructed.

Q3 Looking at one rectangle, what is the formula for the area using h and the function's values?

Q4 Visually, how does it compare with the trapezoid or other rectangle tools? Press the *Show Area* button and calculate the percentage error using this tool. Save this answer.

Q5 Compare the various methods and the area approximations and percentage errors that you found back in Q8 and then in Q2 and Q4 above. If you had to make the tool, which would be the easiest to construct? Which is the most accurate? The least accurate?

The Trapezoid Tool

Name(s): ______________________

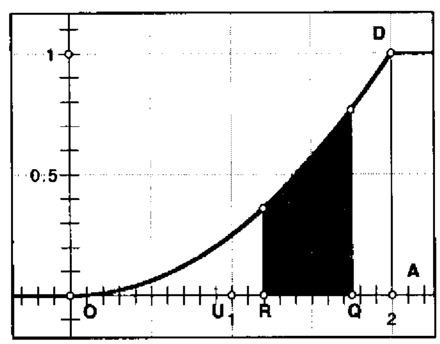
Using trapezoids and rectangles to approximate the area under a curve is a relatively easy way to go. The hard part is really not hard, just time-consuming. Subdividing your region into smaller regions, then constructing the trapezoids or rectangles for every single piece and measuring their areas—that can get tedious. In the last activity, you had a tool that easily made trapezoids and measured their area all at once. Building tools is what this activity is all about.

Sketch and Investigate

1. **Open** the document **TrapezoidTool.gsp** in the **Exploring Integrals** folder. You will be looking at the region on the interval [0, 2].
2. Using the **Point** tool, construct two points on the x-axis on the interval [0, 2]. Label the left point R and the right point Q.
3. Measure their x-coordinates by selecting point R and point Q, and then choosing **Abscissa (x)** from the Measure menu.

Now you need the top of your trapezoid to be anchored on the function at $f(x_R)$ and $f(x_Q)$.

4. Calculate the function values $f(x_R)$ and $f(x_Q)$ by choosing **Measure | Calculate,** then select the expression for f and the measurement x_R to enter them into the calculator. Do the same for point Q.
5. Select x_R and $f(x_R)$ in that order and choose **Graph | Plot as (x, y).** Plot the point $(x_Q, f(x_Q))$ as well.



If you want to, you can also construct the sides by choosing the **Segment** tool and then selecting the two points of each side.

6. Select the four points of the trapezoid and choose **Construct | Quadrilateral Interior** to construct the interior of the trapezoid.

Now you have the actual trapezoid, but you need an area measurement as well. The base is the distance between point R and point Q, $x_Q - x_R$. The heights are the y-values, $f(x_R)$ and $f(x_Q)$.

7. Calculate the trapezoid's area by choosing **Measure | Calculate** and entering $0.5 \cdot (x_Q - x_R) \cdot (f(x_R) + f(x_Q))$. (Why does this formula work?)
8. Label this measurement *AreaT* and check Use Label In Custom Tools on the Label panel.

To make a tool, you need to choose the objects that are necessary to construct your final goal and you need to select the objects you want to end up with.

If you constructed the line segments in step 6, select each of those as well.

9. Select the givens: point *R*, point *Q*, and the expression for $f(x)$. Then select the results: the region's interior, the points on the function, and measurement *AreaT*. Choose **Create New Tool** from **Custom** tools and name this tool **Trapezoid.** Check Show Script View and click OK.

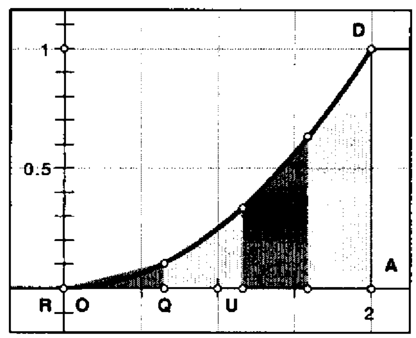
After you're done, function *f* will be assumed.

10. In the tool's Script View, double-click given Function *f* and check Automatically Match Sketch Object. Click OK.

Choose **Edit | Undo Trapezoid** until you are back to your original trapezoid.

Now you have a trapezoid tool, so all you need to do to make a new trapezoid is choose your tool and make two points on the *x*-axis. Try it! When you're done experimenting, undo all trapezoids except for your original one.

11. Drag point *R* to the origin and point *Q* anywhere between 0 and 1.
12. Choose **Trapezoid** from **Custom** tools. Match point *Q* and click on the axis to construct a new point anywhere on the *x*-axis to the right of point *Q* except the unit point—point *U*.
13. Continuing with your **Trapezoid** tool, match the new point you constructed in step 12, and construct a new point anywhere on the *x*-axis to the right of that point. (Make each new trapezoid a different color.)



14. Do step 13 one more time, matching the previous point and then constructing a new point to the right of that, but this time, do *not* match point *A*. (You want the freedom to move your trapezoids, and point *A*, like point *U*, cannot move.)
15. You now have four trapezoids linked together. Move the points on the *x*-axis so that these four trapezoids completely cover the interval [0, 2].
16. Choose **Calculate** from the Measure menu and sum all the trapezoid measurements, *AreaT*. Label this measurement *Approximation.*

Q1 What is your approximation for the total area using four trapezoids?

17. Press the *Show Area Tools* button, then the *Reset P* button, and then the *Calculate Area* button to calculate the actual area under the curve and to shade in that region.

Q2 What is the actual area under the curve from $x = 0$ to $x = 2$?

18. Calculate your approximation percentage error:

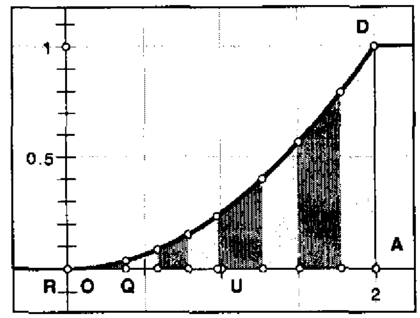
$$\%Error = \frac{|Actual\ Area - Approximation|}{Actual\ Area}$$

Q3 Move the points on the x-axis to find the four trapezoids that give the smallest percentage error. What x-values did you find?

Exploration 1

Check the status line to make sure that you have not selected point P. If you have, click on the point again.

1. Erase traces and slide your four trapezoids to the left so they are all *inside* the region from $x = 0$ to $x = 1$, and then use your **Trapezoid** tool to construct four more trapezoids, linking them as before.



2. Once you have eight trapezoids, move them so that they cover the interval from $x = 0$ to $x = 2$.
3. With the **Arrow** tool, double-click on the measurement *Approximation*, and add the four new *AreaT* measurements. (So *%Error* will automatically update itself.)

Q1 What is your approximation for the area using eight trapezoids?

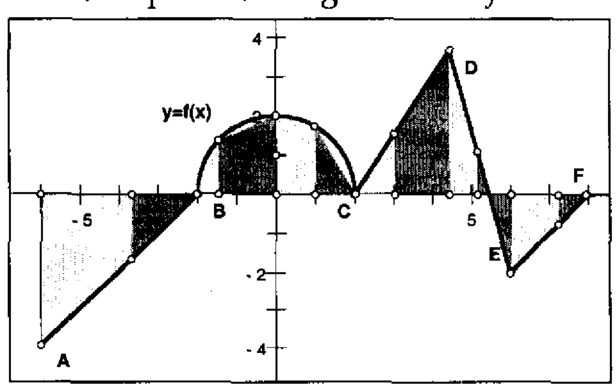
Q2 Move the points on the x-axis to find the eight trapezoids that give the smallest percentage error. What x-values did you find?

Exploration 2

Go to page 2 and experiment with your **Trapezoid** tool on this page. In particular, try each of the following constructions.

A. Construct at least one trapezoid in an interval where the function is below the x-axis during that whole interval.

B. Construct at least one trapezoid with one point in a region where f is below the x-axis ($f < 0$) and the other point in a region where f is above the x-axis ($f > 0$).



Q1 What happens when you use your **Trapezoid** tool on a part of the curve that is entirely below the x-axis?

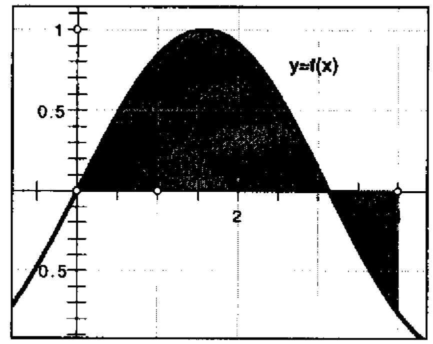
Q2 What happens when you follow the instructions in B?

Q3 Cover the region with trapezoids to get an approximation for the total area under the curve, and then compare it with the "real" area using the **Area** tools. What values did you get?

Accumulating Area

Name(s): ____________________

How would you describe the shaded region shown here? You could say: The shaded region is the area between the x-axis and the curve $f(x)$ on the interval $0 \le x \le 4$. Or, if you didn't want to use all those words, you could say: The shaded region is



$$\int_0^4 f \quad \text{or} \quad \int_0^4 f(x)\,dx$$

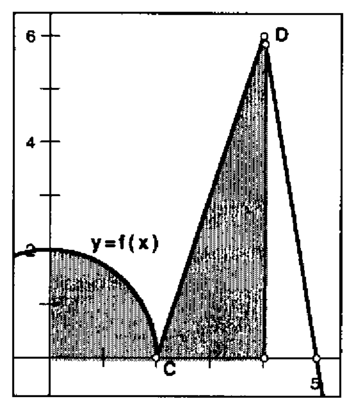
which is much faster to write!

In general, the notation $\int_a^b f(x)\,dx$ represents the *signed area* between the curve f and the x-axis on the interval $a \le x \le b$. This means that the area below the x-axis is counted as negative. This activity will acquaint you with this notation, which is called the *integral,* and help you translate it into the signed area it represents.

Sketch and Investigate

1. **Open** the document **Area2.gsp** in the **Exploring Integrals** folder. You have a function f composed of some line segments and a semicircle connected by moveable points.

If you need to evaluate the integral $\int_0^4 f(x)\,dx$, the first step is to translate it into the language of areas. This integral stands for the area between f and the x-axis from $x = 0$ to $x = 4$, as shown. This area is easy to find—you have a quarter-circle on $0 \le x \le 2$ and a right triangle on $2 \le x \le 4$.



So on [0, 2] you have

$$\int_0^2 f(x)\,dx = 0.25\pi r^2 = 0.25\pi(2)^2 = \pi$$

and on [2, 4] you have

$$\int_2^4 f(x)\,dx = 0.5(base)(height) = 0.5(2)(6) = 6$$

so $$\int_0^4 f(x)\,dx = 6 + \pi$$

2. To check this with the **Area** tools, press the *Show Area Tools* button.

There are three new points on the x-axis—points *start, finish,* and P. Points P and *start* should be at the origin. Point P will sweep out the area under the curve from point *start* to point *finish*. Point P has not moved yet, so the measurement *AreaP* is 0.

3. Press the *Calculate Area* button to calculate the area between *f* and the *x*-axis on the interval [*start, finish*] and to shade in that region.

Q1 Is the value of the measurement *AreaP* close to $6 + \pi$? Why isn't it exactly $6 + \pi$ or even 9.142?

Q2 Based on the above reasoning, evaluate $\int_{-2}^{4} f(x)\,dx$.

Before you move the point, check the status line to make sure you have selected the right point. If you haven't, click on the point again.

4. To check your answer, move point *start* to point *B*, press the *Reset P* button, and then press the *Calculate Area* button.

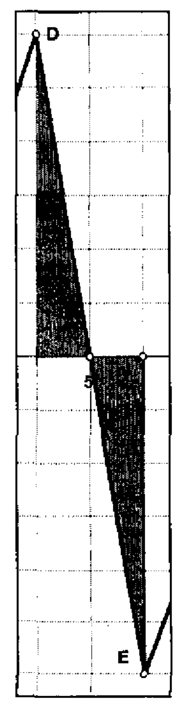
Q3 What do you think will happen to the area measurement if you switch the order of the integral, in other words, what is $\int_{4}^{-2} f(x)\,dx$?

5. To check your answer, move point *start* to $x = 4$ and point *finish* as close as you can get to $x = -2$, then press the *Reset P* button.

6. Choose **Erase Traces** from the Display menu and then press the *Calculate Area* button.

Q4 What is the area between *f* and the *x*-axis from $x = 4$ to $x = -2$?

Now, what happens if your function goes below the *x*-axis? For example, suppose you want to evaluate $\int_{4}^{6} f(x)\,dx$.



Q5 Translate the integral into a statement about areas.

Q6 What familiar geometric objects make up the area you described in Q5?

The grid is shown here for comparison. It doesn't appear in the sketch.

Q7 Using your familiar objects, evaluate $\int_{4}^{6} f(x)\,dx$. (*Hint:* You can do this one quickest by thinking.)

7. Make sure point *start* is at $x = 4$ and move point *finish* to $x = 6$. Press the *Reset P* button.

You can also erase traces by pressing the Esc key twice.

8. Choose **Erase Traces** from the Display menu, and then press the *Calculate Area* button to check your answer. Does the result agree with your calculation?

Q8 Evaluate $\int_{-6}^{-3} f(x)\,dx$ using the process in Q5–Q7 and check your answer using steps 7 and 8.

If you fix your starting point with $x_{start} = -6$, you can define a new function, $A(x_P) = \int_{-6}^{x_P} f(x)\,dx$, which accumulates the signed area between *f* and the *x*-axis as *P* moves along the *x*-axis.

Q9 Why is $A(-6) = \int_{-6}^{-6} f(x)\,dx = 0$?

Q10 What is $A(-3)$?

To get an idea of how this area function behaves as point P moves along the x-axis, you'll plot the point $(x_P, A(x_P))$ and let Sketchpad do the work.

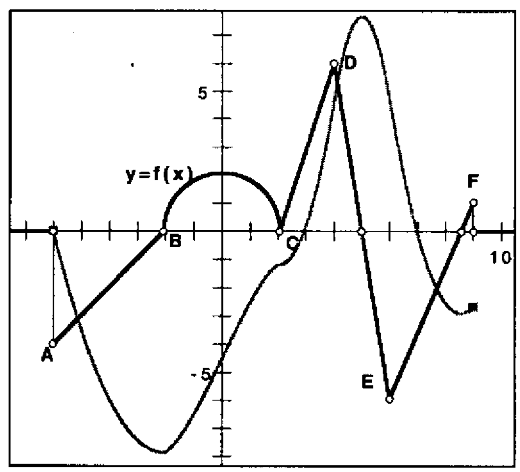
9. Move point *start* to $x = -6$ exactly. Now move point *finish* to $x = 9$. The measurement *AreaP* is now the function $A(x_P) = \int_{-6}^{x_P} f(x)\,dx$.

To turn off tracing, choose **Trace Segment** from the Display menu.

10. Select the line segment that joins point P to the curve. Turn off tracing for the segment. Erase all traces.

If you can't see the new point, scroll or enlarge the window until you do. To enlarge the window, press the *Show Unit Points* button, resize the window, then press the *Set Function* button.

11. Select measurements x_P and *AreaP* in that order and choose **Plot as (x, y)** from the Graph menu.
12. Give this new point a bright new color from the Color submenu of the Display menu. Turn on tracing for this point and label it point I.
13. Press the *Reset P* button and then the *Calculate Area* button to move point P along the x-axis and create a trace of the area function.



Q11 Why does the area trace decrease as soon as point P moves away from point *start*?

Q12 Why doesn't the trace become positive as soon as point P is to the right of point B?

Q13 What is the significance (in terms of area) of the trace's first root to the right of point *start*? The second root?

Q14 What is significant about the original function f's roots? Why is this true?

14. Turn off tracing for point I and erase all traces.

Check the status line to see that point P is selected. If not, click on the point again.

15. Select points P and I and choose **Locus** from the Construct menu.

The locus you constructed should look like the trace you had above. The advantage of a locus is that if you move anything in your sketch, the locus will update itself, whereas a trace will not.

Be sure to keep points A, B, C, D, E, and F lined up in that order from left to right. If point C moves to the right of point D, the line segment CD will no longer exist.

There are quite a few familiar relationships between the original function f and this new locus—including the ones suggested in Q11–Q14. See if you can find some of them by trying the experiments below.

A. Move point B (which also controls point C) to make the radius of the semicircle larger, then smaller.

B. Press the *Set Function* button to move point B back to (–2, 0). Now move point A around in the plane. (Make sure to stay to the left of point B.) Try dragging point A to various places below the x-axis, and then move point A to various places above the x-axis.

C. Press the *Set Function* button to move point A back to (–6, –4). Now move point D around in the plane. (Make sure to stay between point C and point E.) Drag point D to various places above the x-axis, and then drag point D to various places below the x-axis.

D. Follow step C with points E and F.

Q15 List the various patterns that you found between the two functions or in the area function alone. How many patterns were you able to find? Any conjectures about the relationship between the two functions?

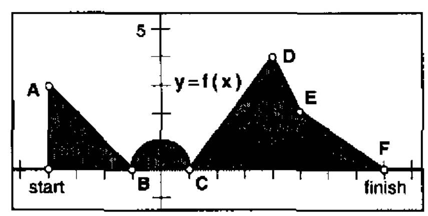
Explore More

Will the area function's shape change if you move point *start* to a value other than $x = -6$?

1. Select point *start* and move it along the x-axis.

Q1 Does the area function's shape change when your starting point is shifted along the x-axis? If so, how? If not, what changes, and why?

Q2 Write a conjecture in words for how the two area functions $\int_{-6}^{x_P} f(x)dx$ and $\int_{x_{start}}^{x_P} f(x)dx$ are related.



Be sure to keep points A, B, C, D, E, and F lined up in that order from left to right. If point C moves to the right of point D, the line segment CD will no longer exist.

2. Make a new shape for your area function by moving one or more points—point A, B, D, E, or F. Then move point *start* again along the x-axis.

Q3 Does your conjecture from Q2 still hold? Write the conjecture in integral notation.

3. Fix point *start* at the origin. Move point P to the left of the origin but to the right of point B.

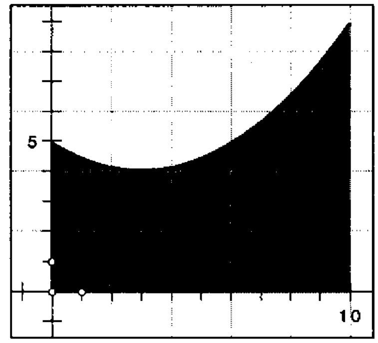
Q4 The following two sentences sound good, but lead to a contradiction. Where is the error?

> The semicircle is above the x-axis from the origin to point P, so the area is positive. Point I, which plots the area, is below the x-axis, so the area is negative.

Area and Integrals

Name(s): ____________________

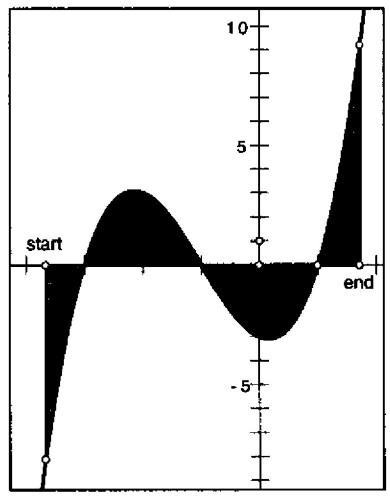
The shaded region in the figure at right is bounded by the x-axis, a function f, and two vertical lines, $x = 0$ and $x = 10$. You have seen that you can approximate the area of such a region with rectangles and trapezoids, and that this area is called the integral. In this activity, you will explore how limits are used to define the integral.



Sketch and Investigate

1. **Open** the document **AreaIntegral.gsp** in the **Exploring Integrals** folder. On page 1 you will find the plot of a function $f(x)$.

2. Press the *Show Integral* button to display the integral of f from point *start* to point *end*. The x-coordinate of point *start* is a, and the x-coordinate of point *end* is b.



In the last activity you learned that the notation for this shaded region, or the integral, is $\int_a^b f$ and that in words we say "the integral from $x = a$ to $x = b$ of the function f." This is just the notation for the integral. The following steps will build the definition.

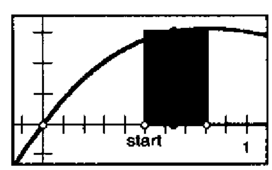
3. Go to page 2 of the document.

4. Choose **Riemann Rectangles** from **Custom** tools. This tool constructs a special type of rectangle that you can use to define the integral.

5. To use this tool, click on point *start*, and then click on the x-axis a little to the right. As you move to make your second click, two points will be constructed. One marks the end of the new segment and a rectangle. The other point, called a *sample point*, is on the base of the rectangle and can be dragged. The tool uses the function's value at this point for the rectangle's height.

You'll actually be clicking on a segment which is on the x-axis.

The label has been left off to avoid clutter.



6. Repeat step 5 four more times, using the right endpoint of the previous rectangle for your first point each time. Finish at point *end*. The widths of these rectangles do not need to be the same.

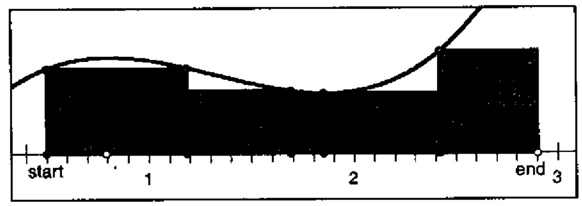
If you end up short of point *end*, drag the last constructed point to point *end* and adjust the others if you'd like.

The tool also calculates the area of each rectangle—the width h times the height $f(x_i)$, where x_i is the sample point of the rectangle. This calculation is the building block of the integral's definition.

The rectangles you have made partition the interval $[a, b]$ into 5 subdivisions, each with area $f(x_i) \cdot h$ for $i = 1$ to 5. The sum of the areas of these rectangles is called a *Riemann sum*. If you place the sample point so that $f(x_i)$ is at the lowest y-value on the subinterval, the Riemann sum is called a *lower sum*.

Here, you will need to drag the light blue sample points on the x-axis, not the points on the function.

7. Position your sample points to create a lower sum by moving each one to the point where the function's value is lowest (see figure).



Click on a measurement to enter it into the calculator.

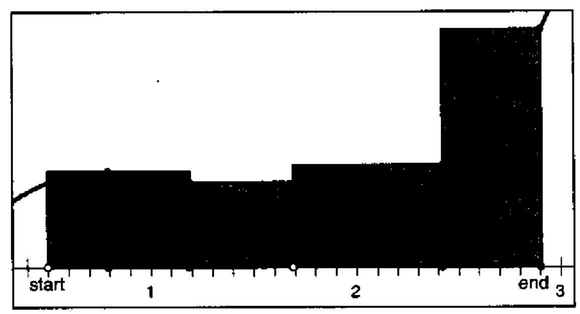
8. Choose **Calculate** from the Measure menu and calculate the lower sum by summing up all five product measurements $f(x_i) \cdot h$. Label this measurement *Sum* with the **Text** tool.

Q1 Write down the value you got in step 8 for the lower Riemann sum. Is it an overestimate or underestimate for the integral from $x = a$ to $x = b$?

An *upper sum* is a Riemann sum where the sample points are positioned so that $f(x_i)$ is the greatest y-value on the subinterval.

Check the status message to make sure you have point i. If you don't, click on that point again.

9. Position your sample points to create an upper sum by moving each sample point to where the function's value is greatest.



Q2 Write down the value for your upper Riemann sum. Is it an overestimate or underestimate for the integral from $x = a$ to $x = b$?

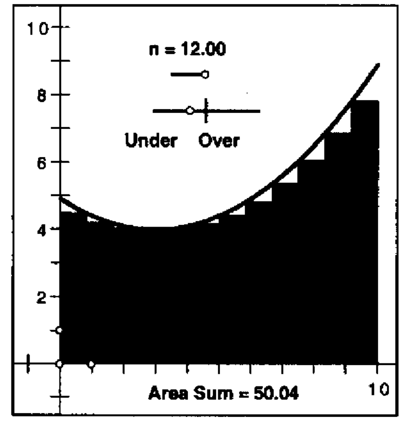
Q3 Why do you think there is a large difference between the lower sum and upper sum values that you found? What could you do to decrease this difference?

10. Go to page 3 of the document. On this page, there is the plot of a function $f(x)$ with rectangles already constructed to calculate two Riemann sums—one that underestimates and one that overestimates the actual area under the curve.

11. Adjust the slider for n so that there are six rectangles. Move the point on the *Under Over* slider so that the rectangles form a Riemann sum that overestimates the area. Then move the point so they form an underestimate for the area.

Q4 Record both Riemann sums for this example with $n = 6$.

12. Adjust the slider for n so that there are 12 rectangles and use the *Under Over* slider to record both Riemann sums for $n = 12$.



You can continue dragging the slider point outside the window, or expand your window to reach $n = 396$.

13. Repeat step 12 for $n = 24, 48, 96$, and 396. Make a chart of the Riemann sums for the different values of n.

If the limit of the Riemann sums exists as you make the largest of your h values approach 0, then the function is said to be *integrable* on the interval $[a, b]$, and this limit is called the *definite integral* of f on the interval $[a, b]$.

Try averaging the two Riemann sums for each n.

Q5 Does it look like the Riemann sums are converging to the same limit in step 13? If so, to what limit? If not, explain why not.

Q6 Check your answer by pressing the *Show Area* button.

So the integral $\int_a^b f(x)dx$ is defined as the limit of the sum of the n areas as h approaches 0 on the interval $[a, b]$, or $\int_a^b f(x)\,dx = \lim_{h_i \to 0} \sum f(x_i) \cdot h_i$.

In this example, n must be at least 2 to create the over-estimate. (Why?)

Q7 Go to page 4 and estimate $\int_1^{10} f(x)\,dx$ using the limit of the Riemann sums as you did in steps 11–13.

With this definition in hand, you can investigate some special properties of integrals.

14. Go to page 5 of the document. Shown is the constant function $f(x) = d$ where $d = 3$.

Q8 To find the value of $\int_{-2}^{3} f(x)\,dx$, you need only one Riemann rectangle. What is the value of this integral? Choose **Riemann Rectangles** from **Custom** tools to check. Click on point *start* and then point *end*.

15. Use the **Riemann Rectangles** tool again, but this time click on point *end* first and then point *start*.

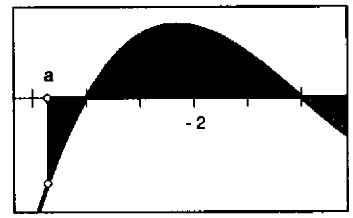
Q9 What happens? What has changed in the calculation of $f(x_i) \cdot h$ to cause this to occur?

Q10 Write an integral that expresses the area that you just found in Q9.

16. Use the sliders to create the function $f(x) \approx -3$.

Q11 What is the value of $\int_{-2}^{3} f(x)\,dx$ for this function? $\int_3^{-2} f(x)\,dx$?

Now that you have done some examples and worked with Riemann rectangles, go back to the function on page 1. Without creating any rectangles, for each of the following integrals, decide whether the integral's value is a positive or negative number and explain how the definition of the integral helps you decide.



Q12 $\int_{-3}^{-1} f(x)\,dx$

Q13 $\int_{-1}^{1} f(x)\,dx$

Q14 $\int_{-3.5}^{-1} f(x)\,dx$

Q15 $\int_{-3}^{1} f(x)\,dx$ if the function $f(x)$ is symmetric around $x = -1$.

Q16 Using your answers above and what you now know about approximating integrals with rectangles and trapezoids, what approximation would you give for the integral of $f(x)$ from $x = -3.5$ to $x = 1.5$ for the function on page 1? Explain.

Explore More

Is it possible to add integrals? For any function f, does

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx?$$

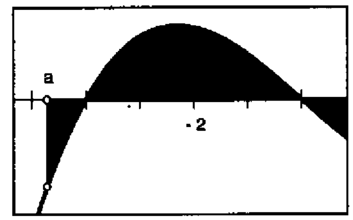
Use the definition of the integral to explain your answer for each case.

Q1 If the value of $\int_1^5 f(x)\,dx = 9$, and $\int_1^5 g(x)\,dx = 3$, would $\int_1^5 (f(x)+g(x))\,dx = 12$? Explain, and sketch two functions that support your answer.

Q2 If the value of $\int_1^5 f(x)\,dx = 9$, would $\int_1^5 2f(x)dx = 18$? Explain and sketch a function that supports your answer.

The Area Function

Name(s): ____________________



In previous activities, you used tools to approximate the area between a curve f and the x-axis from $x = a$ and $x = b$. By finding the limit of your approximations, you calculated the integral —the actual signed area—with a high degree of accuracy. But what if you wanted to find the signed area from $x = a$ to $x = b$ where $b = -4, -3, -2, \ldots$? Calculating all these limits, one at a time, would take forever. What we really need is a function that calculates the integral of f from $x = a$ to any point on the x-axis. Does such a function exist?

Sketch and Investigate

1. **Open** the document **AreaFunction.gsp** in the **Exploring Integrals** folder. On page 1 you will find the plot of a function $f(x)$, point *start* $(a, 0)$, and point *end* $(b, 0)$, where a and b are parameters.

Suppose that you want to find the value of the integral of f from $a = -4$ to any point $x = x_P$, or $\int_{-4}^{x_P} f(x)\,dx$. Does this expression define a function?

Double-click on the point with the **Text** tool to change its label.

2. With the **Point** tool, construct a point on the x-axis. Choose **Measure | Abscissa (x)** to measure the x-coordinate. Label the point P.
3. Choose **Measure | Calculate** and calculate the value of $f(x_P)$.

Select x_P and $f(x_P)$, in that order. Choose **Graph | Plot As (x, y)**.

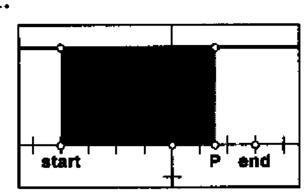
4. Plot the point $(x_P, f(x_P))$.

In this case, the integral $\int_{-4}^{x_P} f(x)\,dx$ is equal to the area of the rectangle from $a = -4$ to $x = x_P$, with width $x_P - a$ and height $f(x_P)$.

5. Calculate this area by choosing **Calculate** from the Measure menu. Label this calculation *Area*(P) with your **Text** tool.

Select the four points in order: point start, point P, $(x_P, f(x_P))$, and $(a, f(a))$.

6. To illustrate the integral, plot the point $(a, f(a))$. Then select the four points that define the rectangle and choose **Construct | Quadrilateral Interior**.



7. Move point P to different places on the x-axis—go left and right of point *start*. Observe the measurement *Area*(P). Can you tell whether this measurement is a function?

The measurement *Area*(P) is the integral of f from $x = a$ to $x = x_P$, but it's hard to tell whether it defines a function just by observing its values. A plot of the values for measurement *Area*(P) would help.

Remember to enter subscripts in brackets.

8. At each point P, where $x = x_P$, we want the y-coordinate to be the accumulated area under f from $a = -4$ to point P, or *Area*(P). So, plot

the point (x_P, *Area*(P)). Label this new point A_P. (If you can't see the new point, drag point P closer to point *start*.)

To make a smoother trace, choose **Display | Animate Point**. Then decrease the speed on the Motion Controller.

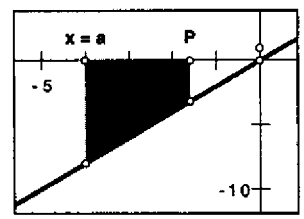
9. With point A_P selected, choose **Display | Trace Plotted Point** to turn on tracing for point A_P. Slowly drag point P along the x-axis to the left and right of point *start*.

Q1 This trace is the plot of $A(x_P) = \int_{-4}^{x_P} f(x)\,dx$. Does the trace look like it defines a function? (Does it pass the vertical line test?) If you move point P to the left of $a = -4$, the rectangular area is above the x-axis, but *Area*(P) is negative. Why?

Q2 Where is $A(x_P)$ zero? Increasing? Decreasing?

Q3 What type of plot is the trace itself? Write an expression for the function that you think the point is tracing. Can you prove that your answer is correct?

10. Go to page 2 to examine $A(x_P) = \int_{a}^{x_P} f(x)\,dx$ for $a = -4$ and a different $f(x)$. Here, f is linear, so the region between the plot of f and the x-axis is not a rectangle, and you will need to calculate the area differently.



See the Activity Notes for an explanation of why you can use the area formula for a trapezoid here.

11. Use the formula for the area of a trapezoid to calculate the area from $a = -4$ to $x = x_P$. Label this calculation *Area*(P).
12. As before, plot the point (x_P, *Area*(P)). Turn on tracing for this point.

Reducing the speed of the animation will trace in more points.

13. With just point P selected, choose **Display | Animate Point** to move point P along the x-axis and to trace the values of the integral $\int_{-4}^{x_P} f(x)\,dx$ as x_P varies.

Q4 Where is $A(x_P) = \int_{-4}^{x_P} f(x)\,dx = 0$? Where is it increasing? Decreasing?

Q5 Describe what happens to the plot of $A(x_P)$ where $f(x) = 0$.

Q6 What type of curve is the trace itself? Write an expression for the function that you think the point is tracing. How can you prove that your answer is correct? (*Hint:* Use the function $f(x) = 2x$ in the formula for the area of the trapezoid.)

Q7 Pick any constant. If you substitute this constant into your expression from Q6, does it give the same value as the area from $a = -4$ to your constant?

In the cases above, you can find an expression that tells you the value of $A(x_P)$ for any value of x_P. To evaluate $\int_{-4}^{b} f(x)dx$, all you need to do is substitute b into this expression. For functions that are not linear or constant, you cannot use a formula from geometry to create the expression, but you can use limits.

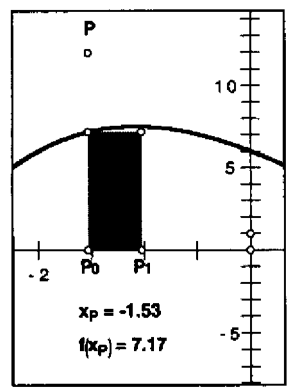
You have built rectangles and calculated the sum of their areas to estimate the integral before. Here, you'll build one rectangle, and use it to trace an approximation of the plot of $A(x_P) = \int_{a}^{x_P} f(x)\,dx$, for $a = -4$.

14. Go to page 3. On this page, you will find a cubic function $f(x)$ and an independent point P. You will use point P to trace the plot of the approximate integral function.

Remember to enter subscripts in brackets.

15. Use the parameter *zero* to plot the point $(x_P, 0)$. Label this point P_0.
16. Calculate $f(x_P)$ and plot the point $(x_P, f(x_P))$.

Again, we need to make a small rectangle so the estimate is fairly accurate. The small adjustable measurement h is used for the width of our rectangle.



17. To make the base of the rectangle, calculate $x_P + h$. Plot the point $(x_P + h, 0)$. Label this point P_1.
18. We want the right side of the rectangle to be the same height as the left side, which is $f(x_P)$. So plot the point $(x_P + h, f(x_P))$. Now you have four points that form a rectangle.

Select the points in clockwise or counterclockwise order.

19. Construct the interior of the rectangle by selecting the four new points and choosing **Construct | Quadrilateral Interior.**
20. Finally, calculate the area of this rectangle, $h \cdot f(x_P)$. Label this calculation *Area*(P).

Now, to sum up, or accumulate, the *Area*(P) values as x_P varies, we need a starting value. We will use the y-coordinate of point P.

To see the accumulation, construct a segment from point P to point Q.

21. From point P_0 to P_1, only the area of the rectangle, *Area*(P), will be accumulated, so calculate the value y_P + *Area*(P). Plot the point $(x_P + h, y_P + Area(P))$ and label this point Q.

Now, you can plot the approximate value of the integral function, $A(x_P) = \int_{a}^{x_P} f(x)\,dx$ where $a = -4$, by using points P and Q.

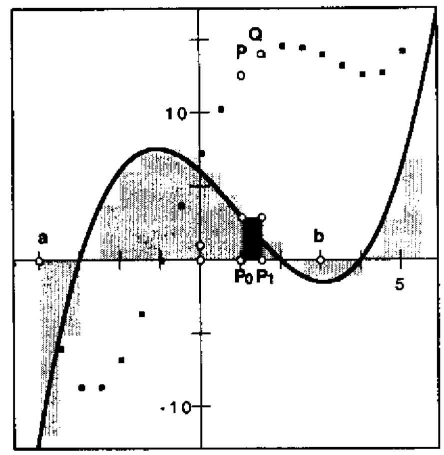
22. Because y_P is the starting value for accumulating the area, if we move point P so that its x-coordinate is $x_P = -4$ then y_P must be 0. (Why?) So press the *Move P→a* button.

Check the status line to make sure you have selected point *P*. If not, click on the point again.

23. The y-value of point *Q* is now the approximate value of $A(x) = \int_a^x f$ at $x = a + h$. To trace the value of this integral as x varies, select point *P*, then point *Q*, and choose **Edit | Action Buttons | Movement**. Choose **instant** speed and click OK.

By tracing the rectangle, you can see if your trace is underestimating or overestimating the integral.

24. Select point *P* and the rectangle, then choose **Display | Trace Objects** to turn on tracing.



25. Press the *Move P→Q* button a few times with $h \approx 0.5$. (If h is negative, the movement will be to the left.)

Q8 Can you tell whether your trace is an overestimate or an underestimate?

Select the *Move* button and choose **Edit | Properties**. On the Move panel, change speed to **instant**. Click OK.

26. To see a more accurate trace, adjust the slider for h to a smaller value, and your *Move P→Q* button to **medium** speed. Press the *Move P→a* button and then press the *Move P→Q* button.

Although this plot is only an approximation of the function $A(x_P) = \int_a^{x_P} f(x)\,dx$, it gives a good idea of the plot of the function. In the next two activities, you will explore how to find an expression for the integral function when you can't use a geometric formula.

Explore More

You can edit the expression for $f(x)$ or the value of the parameter a to create traces of $A(x_P) = \int_a^{x_P} f(x)\,dx$ for other functions and values of a.

Go to page 4 and experiment with tracing the integral approximation for values less than $x = a$ by adjusting the *Direction* slider to –1 and pressing the *Move P→a* button. Try tracing the integral function on both sides of a for each of the combinations below.

Q1 Try $f(x) = 1/x$ for $a = 1$. Be careful as point *P* approaches the discontinuity in the graph of f. Do you recognize this function?

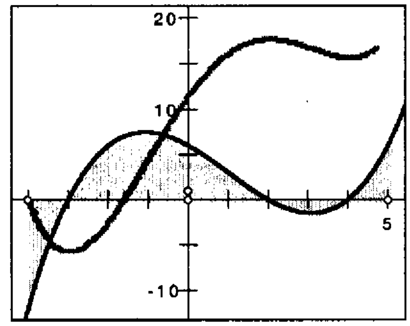
Q2 Try $f(x) = x^2$ for $a = 0$. Do you recognize this function?

Q3 Try $f(x) = e^x$ for $a = 0$, as well. What is this function?

Plotting the Integral

Name(s): ____________________

In the previous activity, you explored the concept of an *area function* by plotting the accumulated signed area from a fixed point ($a = -4$) to a varying value x_P.



In this activity, you will investigate the properties of functions *defined* by integrals. Along the way, you will explore how to find expressions for these functions and learn how to sketch them using paper and pencil.

Sketch and Investigate

1. **Open** the document **PlotIntegral.gsp** in the **Exploring Integrals** folder. On page 1 of this document you will find the plot of a function f and some measurements.

To evaluate $\int_a^{x_P} f$ for any values of a and x_P, it would be convenient to have an expression into which you could substitute values directly. Coming up with such an expression that works for all values of a and x_P requires some experimenting.

All of these measurements and objects were constructed in the previous activity.

2. If f is constant, then $\int_a^{x_P} f$ is equal to the area, labeled *Area*(P), of the rectangle from $x = a$ to $x = x_P$. Point A_P has coordinates (x_P, *Area*(P)). Move point P to see the trace of the integral function for the given a.

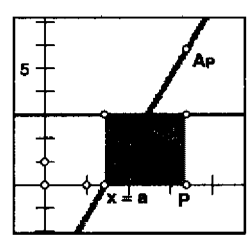
3. To create a dynamic plot of the integral function for different values of a, select points A_P and P. Choose **Locus** from the Construct menu.

To change the value of a, move the point on the x-axis where $x = a$.

4. You now have a plot of $A(x_P) = \int_a^{x_P} f$ where both a and x_P can vary. Experiment with moving point P, then with changing the value of a.

Q1 For any value of a, where is the integral function zero? Increasing? Decreasing?

Q2 What happens to the locus when you move point P? When you change the value of a?



Q3 Write an expression for this locus for all values of a. (Your expression will involve x_P and a.)

Q4 *Challenge:* Press the *Show c* button and then adjust the slider for c. Describe what happens to both $f(x)$ and the locus. Then write an expression for the locus in terms of a, c, and x_P.

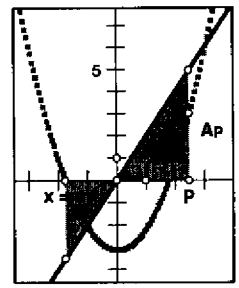
Now let's see what happens for a different function.

See the previous activity to learn how the objects here were constructed. The trapezoid formula was used for *Area*(*P*) here.

5. Go to page 2 of the document. Here f is a linear function. Again, point A_P plots the point (x_P, *Area*(*P*)) where *Area*(*P*) is the value of the integral from $x = a$ to $x = x_P$, and both a and x_P can vary.

6. Select points A_P and P, then choose **Locus** from the Construct menu.

7. You now have a plot of $A(x_P) = \int_a^{x_P} f$ where both a and x_P can vary. Experiment with moving point P, then with changing the value of a.



Q5 For any value of a, where is the integral function zero? Increasing? Decreasing?

Q6 What happens in your sketch when you move point P? When you change the value of a?

Q7 Write an expression for this locus for all values of a. (Your expression will involve x_P and a.)

Another challenge: Can you answer Q4 for this function?

Q8 *Challenge:* In both examples above, varying a resulted in a family of vertically translated functions. Can you determine the distance between the plots for two different values of a?

Once you go beyond linear functions, you can't use a geometric formula for *Area*(*P*). Instead, we will use approximations and limits.

See the previous activity for the construction steps or for more explanation.

8. Go to page 3. Here you will find a plot of $f(x) = \sin(x)$, points P and Q, and a trapezoid. The trapezoid approximates the integral of f on the interval $x = a$ to $x = a + h$. You will use these objects to construct a dynamic approximation of $A(x) = \int_a^x f$ using iteration.

9. The difference between the y-coordinates of points P and Q is the signed area of the trapezoid. Move point P and track how point Q moves in response to the changes in the trapezoid and its area.

As in the previous activity, the y-coordinate of point P will be the starting point for accumulating the area.

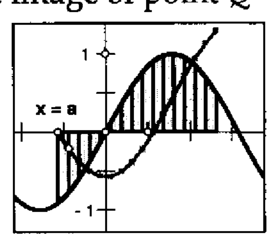
10. Because $A(x) = \int_a^x f = 0$ at $x = a$, merge point P with the point at $x = a$. To do this, select point P and then the point where $x = a$. Choose **Merge Points** from the Edit menu.

Point P is now the point where $x = a$.

11. Select the point where $x = a$ and choose **Transform | Iterate** and then click on point Q for the First Image. Uncheck Tabulate Iterated Values from the Structure pop-up menu, then press **Iterate.** The images of three more trapezoids will be constructed. Also, the image of point Q will create an approximate plot of $A(x)$.

To select an iterated image, click once on any of the newly constructed images.

12. Select the iterated image of the trapezoids and press the plus (+) key to increase the number of iterations and see more of the plot.



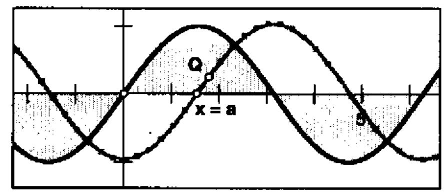
Q9 For this approximate plot of $A(x) = \int_a^x f$, where is $A(x) = 0$? Increasing? Decreasing?

Q10 What will the plot of $A(x)$ look like for values of x less than a? Sketch your prediction in the margin.

Q11 Go to page 4 to see the answer to Q10. Move around the point where $x = a$. Describe the family of curves for $A(x)$, for varying a.

A basic function is one that has no transformations— x and $\sin(x)$ are basic functions; $x + 3$, $5x$, $\sin(3x)$, and $\sin(x) + 4$ are not.

Q12 Does the plot of $A(x)$ look familiar? Write down which basic function you think is being approximated here.

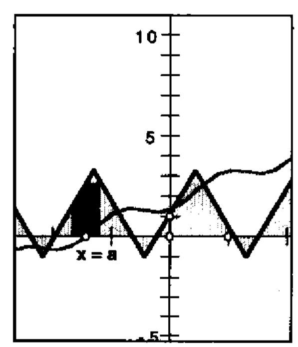


13. Choose **Graph | Plot New Function.** Enter your prediction from Q12. Does your plot match the iterated image's shape for all values of a (ignoring translations)?

Q13 *Challenge:* Now try to match the plot exactly, including the translation. Double-click on your prediction to edit the expression. Can you determine the role of a in the expression that will match the plot?

You have looked at three different types of functions and predicted very specific expressions for the integral function $A(x) = \int_a^x f$. Now let's look for some general relationships. Look back over pages 1–4 and use page 5, as well, to answer these questions and sum up the relationship between f and $A(x)$.

Q14 What can you say about $A(x) = \int_a^x f$ if



a. f is positive? f is negative?

b. f changes sign?

c. f is increasing?

d. f is decreasing?

e. f has a maximum or minimum?

Q15 What can you say about f when

a. $A(x)$ is increasing?

b. $A(x)$ is decreasing?

c. $A(x)$ has a maximum or minimum?

d. $A(x)$ is translated vertically?

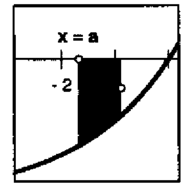
Are the connections between the plots of f and $A(x)$ familiar? Make a conjecture about the relationship between f and $A(x)$ based on your answers to Q14 and Q15.

Explore More

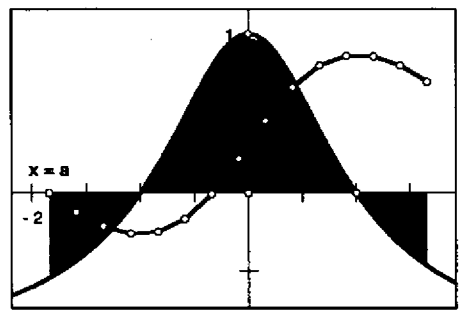
Earlier, you constructed an approximate plot of the integral using iteration. Here, to gain more experience sketching a plot of the integral function by hand, you will use a tool that mimics how you might draw it yourself with paper and pencil.

Go back to steps 8 and 9 if you have forgotten what these objects represent.

1. Go to page 6. Choose **Plot Area with Trapezoids** from **Custom** tools. With the tool click once on the point where $x = a$ to create a trapezoid and a segment.



This tool will automatically match the function f and the parameter h and create a trapezoid between the function plot and the x-axis. It will also construct two points whose y-values differ by the signed area of the trapezoid and the segment between these two points.



2. Using the tool again, click on the right endpoint of the segment constructed in step 1. This will add the area of the next trapezoid to the first one.
3. Continue using the tool to build the approximate plot of the integral function, each time using the right endpoint of the previously constructed segment.
4. Choose **Plot Area with Trapezoids(left)** from **Custom** tools. Click again on the point where $x = a$ to extend your plot to the left. Continue using this tool, each time clicking on the left endpoint of the previous segment.

Q1 When will the constructed segment have a positive slope? A negative slope?

Q2 How can you tell whether the next constructed segment will be sloped at a greater or lesser angle than the previous segment?

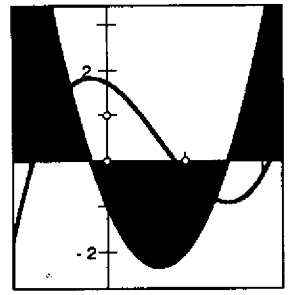
Q3 When will the plot you make have a maximum? A minimum?

Go to page 7 and sketch the plot of $A(x) = \int_a^x f$ for the given function in the margin, using your answers from Q1–Q3 above. Then check your answer using the tool, step by step. How did you do?

Getting Down to Fundamentals

Name(s): ____________________

In the last activity, you explored properties of the function $A(x) = \int_a^x f$ for any a. The relationships that you found between the plots of f and $A(x)$ hinted that $A(x)$ is an antiderivative of $f(x)$. In this activity, you will examine this possibility in a variety of ways.



Sketch and Investigate

First, let's see if a plot of $A(x) = \int_a^x f$ matches a symbolic antiderivative.

1. **Open** the document **Fundamentals.gsp** in the **Exploring Integrals** folder. On page 1 of this document you will find the plot of the function $f(x) = 3x^2 - 4x - 1$ and some sliders.

Click on measurement C to enter it into the function editor.

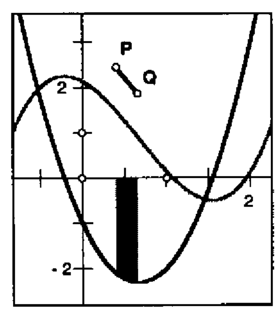
2. Choose **Plot New Function** from the Graph menu, and enter the antiderivative of $f(x)$, $x^3 - 2x^2 - x + C$.

Leave the plot's thickness as **Thin**.

3. Double-click on the equation for the antiderivative with your **Text** tool and label it F.
4. Press the $C \to 2$ button so that f has an intercept at $x = -1$.

Now you will use a tool from a previous activity to trace a plot of the integral function $A(x)$ and see if the trace is identical to the plot of $F(x)$.

5. Choose **Area with Trapezoids** from **Custom** tools. This tool will automatically match the function $f(x)$ and the parameter h.
6. With the tool, click in an empty spot in the plane. With the **Text** tool, label this point P. Label the other point Q.



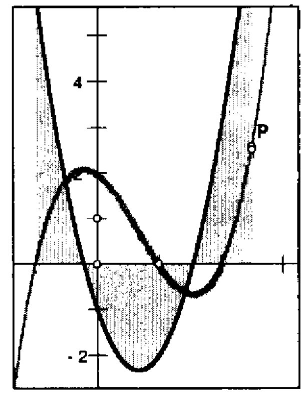
This tool also creates a trapezoid whose area approximates the integral of $f(x)$ on $[x_P, x_Q]$. Now, you can create an approximated trace of $A(x)$ with these two points by having point P *chase* point Q.

Select point P, then point Q, and choose **Edit | Action Buttons | Movement.**

7. Make a button that will move point P to point Q at **medium** speed.
8. Move point P to $(-1, 0)$. Then select point P and the trapezoid and choose **Trace Objects** from the Display menu.
9. Press the *h→0.01* button so you will get a very close approximation of $A(x)$ when you create the trace.

If point P moves off your screen, choose **Edit | Undo Animate Point.**

10. Then press the *Move P→Q* button to create an approximate trace of the integral function.



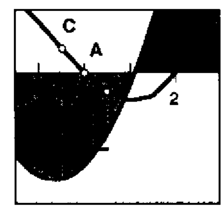
Your trace of $A(x) = \int_a^x f$ and the plot of $F(x)$ should be almost identical. Try this again. Adjust the slider for C to any value, then move point P to any point on the plot of F, erase your traces, and then press the *Move P→Q* button. Does it work again?

To erase a trace, press the Esc key twice.

Is $A(x) = \int_a^x f$ really an antiderivative of f? If it is, then $A'(x)$ must be $f(x)$. Let's see if this is the case.

See the previous activity to learn how to construct $A(x)$ using iteration.

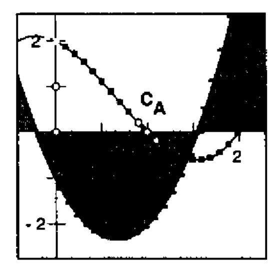
11. Go to page 2. Here you will find a plot of an approximation of $A(x)$ constructed using iteration. Although this plot is only an approximation, you can still use it to look for evidence that its derivative is $f(x)$. We'll test this idea graphically by plotting the slope of some secants to the plot of $A(x)$, and then taking a limit.



12. Choose **Secant Slope** from **Custom** tools. With the tool, click on points A and B. This tool creates a horizontal segment—a "step"—with y-value equal to slope AB, on the interval $[x_A, x_B]$.

Press the Hide Area button before performing the iteration to avoid creating unnecessary objects.

13. With the **Arrow** tool, select point A for the pre-image point and choose **Transform | Iterate**. For the first image, click on point B. Uncheck **Tabulate Iterated Values** in the Structure pop-up menu. Press the plus (+) key until **Number of Iterations** is 10. Click Iterate.



14. Select just point A again for the pre-image point and choose **Transform | Iterate**. This time, for the first image, click on point C. Uncheck **Tabulate Iterated Values** in the Structure pop-up menu. Press the plus (+) key until **Number of Iterations** is 10. Click Iterate.

15. Adjust the slider for h to a small value to approximate the limit.

Q1 Explain what happened graphically in step 15 at point A and to the "steps." How does this add evidence that $A'(x) = f(x)$ at point A?

Q2 If $A'(x) = f(x)$, then what is the value of $\lim_{x_B \to x_A} \frac{y_B - y_A}{x_B - x_A}$?

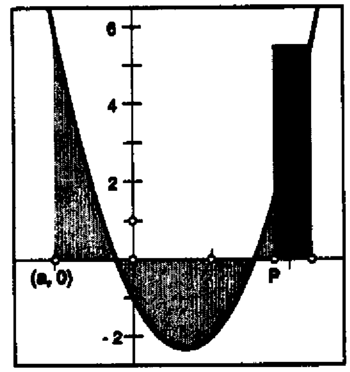
16. Test your answer to Q2. Make $h \approx 0.25$. Select segment AB and choose **Measure | Slope.** Then use the slider to adjust h and make point B approach point A. How did you do?

From the steps above, it looks like the slope of $A(x)$ is the y-value of $f(x)$ at point A. In fact, each iterated "step" intersected the plot of f. To prove that

$A'(x) = f(x)$ algebraically, we can apply the definition of the derivative to the function A.

Now the question is: Does $\lim_{h\to 0} \frac{A(x+h)-A(x)}{h}$ equal $f(x)$? You can answer this question by looking at this limit at an arbitrary point.

17. Go to page 3. Here you will see the integral of f from $x = a$ to $x = x_P$ shaded in light green. This area is equal to $A(x_P)$.



Q3 Press the *Show Locus* button to see the integral from $x = x_P$ to $x = x_P + h$. Explain why this integral equals $A(x_P + h) - A(x_P)$.

You can move point *P* to see how the areas change.

18. Hide this locus and then press the *Show Rectangles* button. One of these rectangles is an overestimate of the value of $A(x_P + h) - A(x_P)$, and the other is an underestimate.

Q4 Write an expression for the left rectangle in terms of f. (*Hint:* What is the rectangle's height? Width?)

Q5 Use the expression from Q4 in place of the numerator of the above limit to show that it equals $f(x_P)$.

Explore More

Above, you showed that $A(x)$ is an antiderivative of f. Here you'll look at this fundamental connection between slope and area in another way.

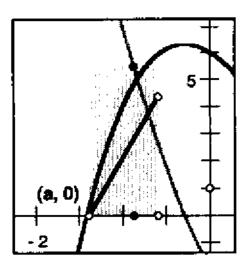
To plot F', select the equation for F' and choose **Graph | Plot Function**.

1. Go to page 4 of the document. Select the function F and choose **Graph | Derivative**. Plot $F'(x)$. Relabel the function F' as f.

You won't see anything created at first.

2. Choose **Riemann Area** from **Custom** tools. With this tool, click on the point $(a, 0)$ to start the area plot.

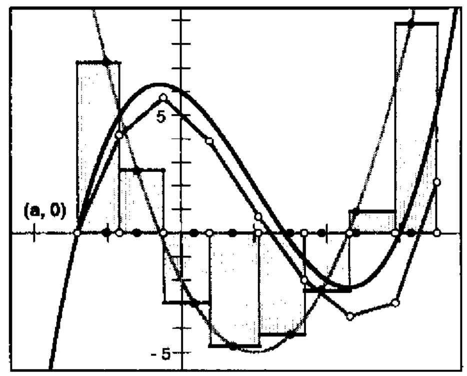
3. Click on the point $(a, 0)$ a second time to start the first rectangle and then click again on the x-axis, but a little to the right of the point $(a, 0)$. As you make this third point, the tool creates a random point between them, called a *sample point*. The tool uses the y-value of $f(x)$ at the sample point for the height of the rectangle and will also create a segment where the difference between the y-coordinates of the endpoints represents the signed area of the rectangle.



After you use the tool once, you only need to select the two points on the x-axis that determine the base of the rectangle. The tool automatically strings together the segments for the plot of accumulated area.

4. Use the tool to make seven more rectangles. Each time, match the first point on the x-axis with the last point of the previous interval. Your intervals do not need to be the same size.

The plot of approximated area probably does not match the plot of $F(x)$ very well, but you can apply a theorem about derivatives to make a better match.



5. First, move the left endpoint of your first rectangle to the x-intercept of $F(x)$ as in the figure at right.

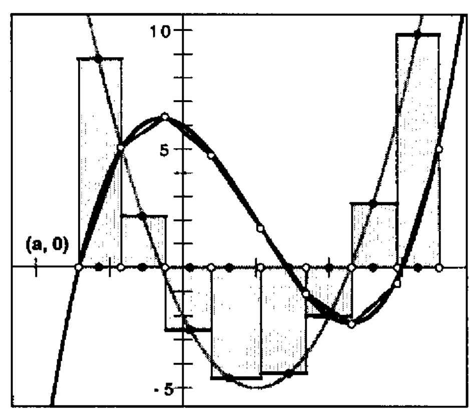
6. Choose **Average Rate** from **Custom** tools. This tool uses points on the x-axis instead of points on the plot and automatically identifies $F(x)$.

You will need to click twice on each interior point. Once to end a segment and once to start the next segment.

7. With the tool, click on the endpoints of each of the intervals you made above. This tool creates horizontal segments, each with y-value equal to the average rate of change of F on its interval.

Q1 What theorem about derivatives is demonstrated at the points where the horizontal segments (the average rates of change of F on the interval) intersect the plot of the derivative? Write out the theorem in its symbolic form.

Now, we'll use these intersections to place the sample points.



Or move the sample point so that the top of the rectangle and the horizontal step are the same.

8. Move the sample point in each interval so that its corresponding point on the rectangle's top is the intersection of the derivative plot and the horizontal segment.

Q2 Use your answer to Q1 to explain why the area of each rectangle is now the *exact* value of the net change in F on each interval, and write an expression for the area of each rectangle in terms of F.

Q3 Explain why the sum of an arbitrary number of rectangles constructed in this way on an interval $[a, b]$ will equal $F(b) - F(a)$. (You can also use limits to prove that the integral of F' on $[a, b]$ equals $F(b) - F(a)$.)

Q4 Explain how to use the ideas in this activity to find the exact value of the integral of a function whose antiderivative you know.

Q5 Given a function f with $f(4) = -1.3$ and the function $g = \int_7^x f$, explain in your own words why $g'(4) = -1.3$.