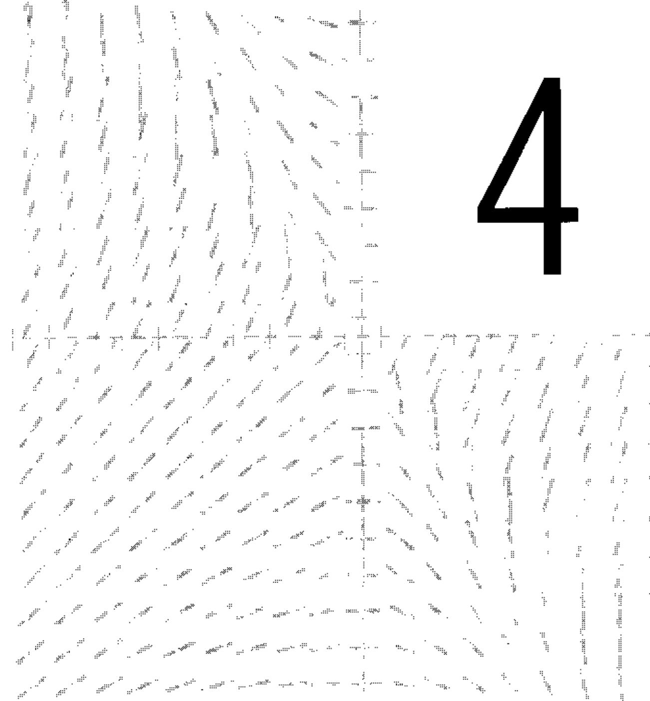
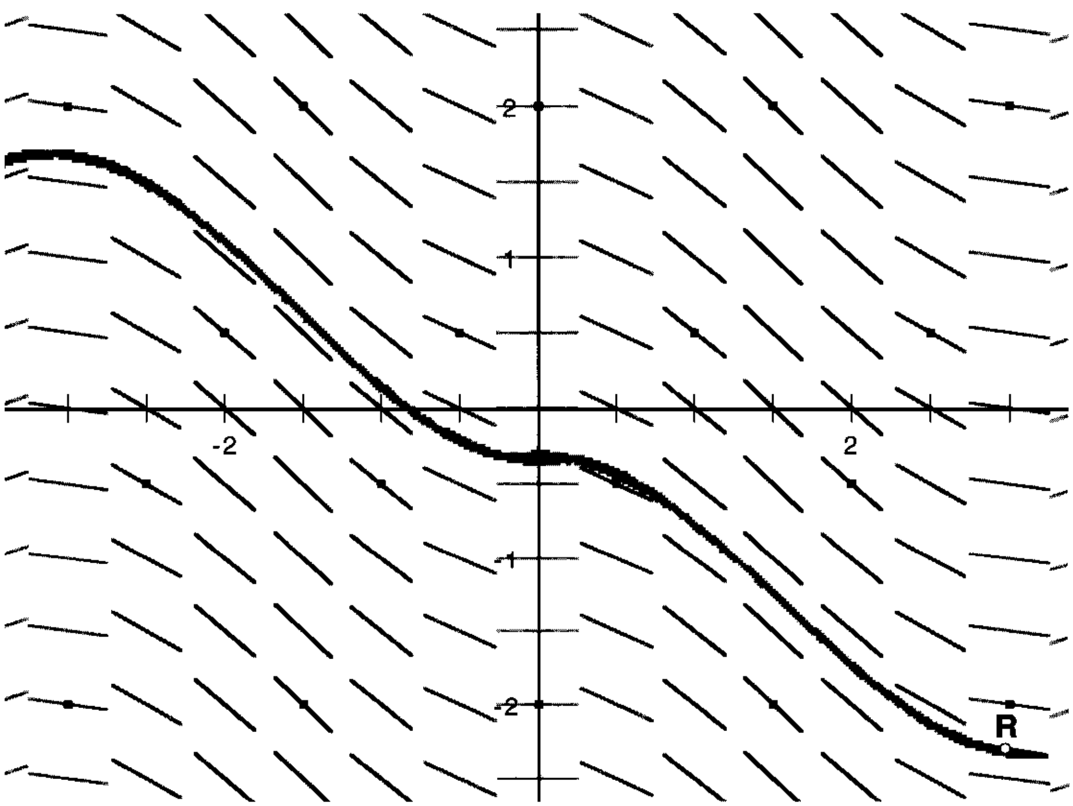


4

Exploring Antiderivatives



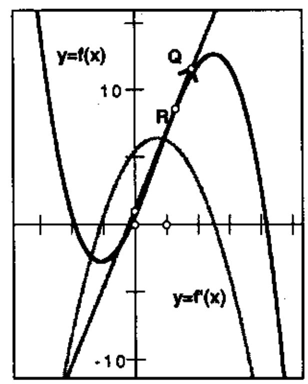
Following Tangent Lines

Name(s): ______________________

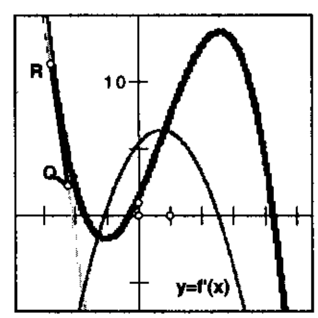
In previous activities, one of your goals was to plot the derivative of a given function f and examine how the two functions f and f' were related. Here, you will again start with a plot, but this time your plot will be the derivative, or slope function, of some unknown function. Your job is to determine the shape of the unknown function when given its derivative. Because you're starting with the derivative and working backward, you're trying to find the *antiderivative* of the function you're given.

Sketch and Investigate

The best tools for predicting what a plot of $f(x)$ might look like when you have the plot of its slope are tangent lines. First, let's review what tangent lines do for us when we *do* have f.



1. **Open** the document **FollowTangent.gsp** in the **Exploring Antiderivatives** folder. You will find a function $f(x)$, its plot, a tangent line to $f(x)$ at point R, and a plot of the derivative of $f(x)$.
2. Move point R along the plot of $f(x)$ and watch how the little arrow at point Q helps show the shape of the plot—even though Q is not a point on the function plot. When you are done observing, move point R as far left as you can. Do you really need f if you have the little arrow at point R?
3. With point R selected, choose **Edit | Split Point From Function Plot** and press the *Hide* $f(x)$ button.
4. Select point R and choose **Trace Point** from the Display menu.



Remember, you already know what f looks like.

5. Can you retrace the now hidden plot of f by moving point R in the direction of point Q? Now that point R is not on the function plot, you'll need to move it carefully in the direction indicated by the arrow. Give it a try!

Q1 Press the *Show* $f(x)$ button. Could you trace out the basic shape of f? What might account for inaccuracies in your trace?

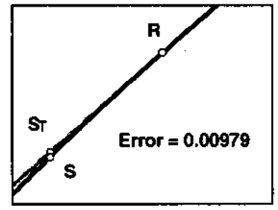
Q2 Write the equation of the tangent line through point R (x_R, y_R) in point-slope form, using derivative notation for the slope.

If you follow the tangent line precisely, you can create a very good rendering of the shape of the function f. However, you also want your prediction of the antiderivative to be fairly accurate. Just how accurate is using the tangent line to approximate values?

6. To review the answer to this question, go to page 2 of the document. Here you have the same function, tangent line, and point R. Point R is fixed this time as the "zoom-in center" at $x = 2$.

> If you want to change the location of point R, edit the parameter x_R.

7. The error you get using the tangent line's y-coordinate instead of the function is represented by the red segment between point S and point S_T. Move point S so that $x_S \approx 1$.



Q3 Use the x- and y-*scale* sliders to zoom in on this function at point R. Compare the y-coordinate that you'd get using the tangent line to the function's y-coordinate. How close do you need to get to $x = 2$ to make your error at most 0.01?

You can see that the tangent line to $f(x)$ at $x = x_R$ is a good approximation of $f(x)$ near R. How is this useful? In the next few steps, you will use the derivative to construct a tangent line to a function without the function being there and use it to sketch the plot of the function.

8. Go to page 3 of the document. Here you will find the plot of a function $f'(x)$ and point R.

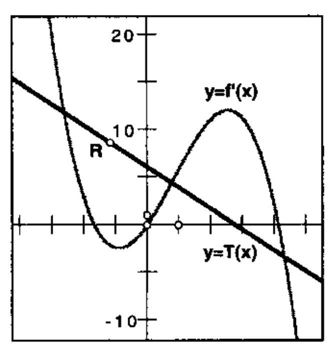
9. You need to construct a tangent line at point R, so measure its x- and y-coordinates.

> Select point R and then choose **Abscissa (x)** from the Measure menu to measure the x-coordinate and **Ordinate (y)** to measure the y-coordinate.

10. If you assume that point R is on $f(x)$, then its y-coordinate is $f(x_R)$. To relabel its y-coordinate as $f(x_R)$, double-click on measurement y_R with the **Text** tool and enter `f(x[R])` on the Label panel.

You do not have the equation or a plot of $f(x)$, but you can still plot its tangent line at point R because you have its slope.

11. Where is this slope? You can get it from the given function $f'(x)$. Choose **Measure | Calculate** and calculate the slope of the tangent line $f'(x_R)$. (What does $f'(x_R)$ represent on the plot of $f'(x)$?)



12. Choose **Graph | Plot New Function** and enter the expression of the tangent line in point-intercept form: $f(x_R) + f'(x_R) \cdot (x - x_R)$. Label this function T.

13. Move point R around the window and observe how the slope of the tangent line changes.

> You also have the value of $f'(x_R)$ so that you can observe the exact value of the slope of the line as you move point R.

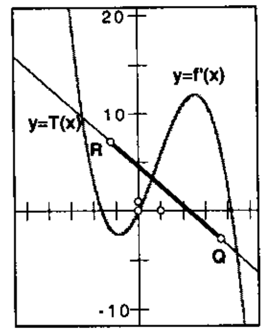
Q4 Where does this line have a negative slope? Where does it have a positive slope? Where does it have a slope of 0?

Q5 Move point R vertically only. What do you observe about the slope of the line? Why does this happen?

> You can move a point vertically using the Up Arrow key.

You have created a line, $T(x)$, whose slope is equal to $f'(x_R)$. While this line has the equation of the tangent line for our unknown function $f(x)$ at point R, it is not actually "tangent" to anything at the moment.

Let's assume for the moment that point R is on our function f. Then, for points near $x = x_R$, this line gives a very close approximation of the plot of $f(x)$. Can you use this line to sketch the plot of the function $f(x)$? Yes! But only if you can follow the tangent line for a short distance. In order to make a small segment on the tangent line, you will use the slider that adjusts the measurement h.

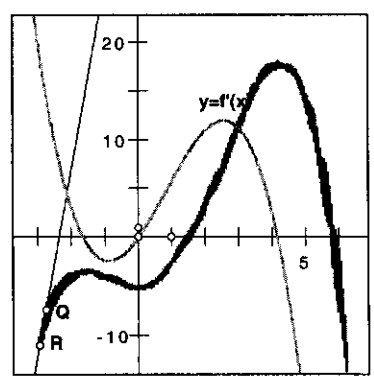


14. Choose **Calculate** from the Measure menu and create the expression $x_R + h$.
15. To plot a second, nearby point on the tangent line, start by calculating the value of $T(x_R + h)$.
16. Deselect the measurement $T(x_R + h)$ and then select the measurements $x_R + h$ and $T(x_R + h)$ in that order and choose **Graph | Plot as (x, y).** Label this point Q.

   Click once in an empty spot to deselect an object.

17. With point Q selected, select point R and choose **Segment** from the Construct menu.
18. With the segment selected, turn on tracing by choosing **Trace Segment** from the Display menu.

Now the segment will leave a trace behind as you move point R around the plane. This segment will trace out the shape of the original unknown function (the antiderivative), if you always move point R in the direction of point Q.



19. Select point R and drag it slowly in the direction of point Q. At the slightest movement of point R, however, the slope of the line will change, so you must constantly adjust, changing the direction in which you move point R as you go. (This may take some practice!)

   If you start out too high or too low on the screen, your trace will run off the edge of the screen. If so, erase the traces and start again in a different position.

20. Once you have practiced this, drag point R to the left of the page so you can trace from left to right. Choose **Display | Erase Traces,** and then carefully move point R, always in the direction of point Q, to trace out the antiderivative.

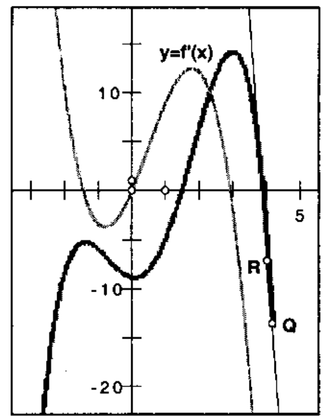
You may have trouble at first tracing in the correct direction. Keep trying, and you'll get better at following the direction of point Q.

Q6 Examine the trace you have created. Where is the function it represents increasing? Where is it decreasing? Where does it reach a local maximum or minimum?

Q7 How do your answers to Q6 compare to your answers to Q4?

The accuracy also depends on h being relatively close to 0. The closer to 0, the more accurate the trace.

The accuracy of the antiderivative you've traced so far depends on your ability to follow the direction of point Q. You can automate this process by having point R try to move toward point Q. This will guarantee that it goes in the correct direction. What's more, point R can never catch up with point Q, so it will keep moving toward it indefinitely.



21. Erase your traces. Select point R and point Q, in that order. Choose **Movement** from the Action Buttons submenu of the Edit menu and keep the speed at **medium**. Click OK.

If you start out too high or too low on the screen, your trace will run off the edge of the screen. If so, erase the traces and start again in a different position.

22. Position point R where you want tracing to start, erase traces and then press the *Move Point* button and watch it go.

How accurate was your trace compared with the "automatic" trace?

23. Choose **Undo Animate Point** from the Edit menu, and point R will return to its initial location.

24. Make several traces, starting from the same horizontal position but from different vertical positions. To do this, turn off tracing for the segment and move point R up or down vertically, then turn on tracing again.

Q8 What do you notice about the traces? Can all these traces be approximations for the antiderivative? If so, how can this be? If not, explain why they are not.

Explore More

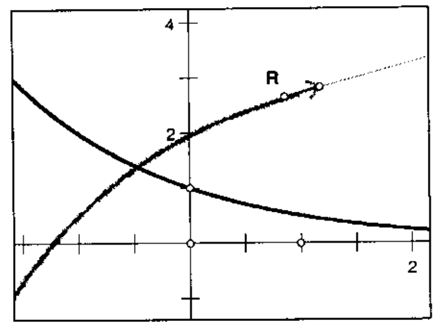
Look at the functions on pages 4–8. Each page has a different kind of function as the given slope function. For each one, make a prediction for the antiderivative, then follow these steps:

1. Drag point R to the left of the page so that you can trace from left to right. Press Esc twice to erase any existing traces, and then carefully move point R, always in the direction of point Q, in order to trace out the antiderivative. Repeat this step until you have a relatively smooth trace that you like.

2. Choose **Edit | Undo Animate Point** to return point R to your initial starting point. Then press the *Move* $R \rightarrow Q$ button. Compare your trace with the automatic trace.

Step to the Antiderivative

Name(s): ______________________



The fact that the tangent line to a function at a given point is a close approximation of the function near that point is a very useful thing. In the previous activity, you used tangent lines to determine the shape of the plot of $f(x)$ given a plot of its derivative, $f'(x)$. In this activity, you'll use tangent lines to construct tools that will create dynamic approximations of the antiderivative, rather than traces. In the process, you'll look rather closely at the construction of the approximate antiderivative so that you can train yourself in plotting antiderivatives by hand.

Sketch and Investigate

1. **Open** the document **StepTangent.gsp** in the **Exploring Antiderivatives** folder. On page 1 you will find a function plot $f(x)$, a few sliders, and an independent point R. Suppose that the function $f(x)$ is the derivative of some unknown function F, so $F'(x) = f(x)$.

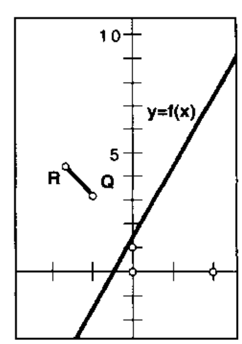
> If you did not do the last activity, go to page 89 for more in-depth explanations.

2. You will also find line $T(x)$ through point R. This line has a slope equal to $F'(x_R)$, or $f(x_R)$—so $T(x) = y_R + f(x_R)(x - x_R)$—and point Q with coordinates $(x_R + h, T(x_R + h))$, just as you built in the last activity.

Q1 Verify that the line $T(x)$ does have slope $F'(x_R)$, or $f(x_R)$, using the appropriate measurements from the plots and point R. Check that your verification holds even if you move point R.

> To trace the location of point R as it follows point Q, you can press the *Move R→Q* button.

For points near $x = x_R$, this line gives a very close approximation of the plot of $F(x)$, and in the previous activity, you approximated $F(x)$ by tracing point R as it followed point Q. This time, you'll construct a tool that creates a tiny segment of that line so you can build an approximate plot of $F(x)$, one segment at a time, simulating how you might sketch an antiderivative with paper and pencil.

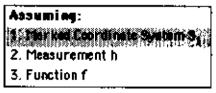


3. Construct a segment from point R to point Q by selecting point R and point Q and choosing **Segment** from the Construct menu.

> To hide point Q's label, click once on point Q with the **Text** tool.

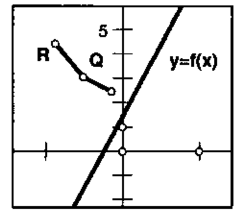
4. Select line $T(x)$ and choose **Display | Hide Line.** Hide point Q's label.
5. To create the new tool, select the givens: point R, measurement h, and the expression for $f(x)$. Then select the results: point Q and the segment RQ. Choose **Create New Tool** from **Custom** tools and name this tool **Antiderivative Segment.** Check Show Script View and click OK.

After you're done, Function f and Measurement h will be assumed as shown.



6. In the tool's Script View, double-click given Measurement h and check Automatically Match Sketch Object. Click OK. Do this again for given Function f.

7. Choose **Antiderivative Segment** from **Custom** tools. Click on point Q.

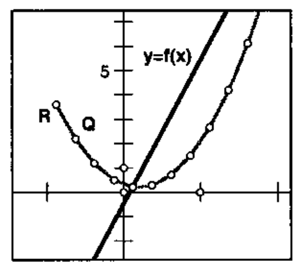


When you clicked on point Q with the tool, you made a new segment that starts at point Q and has a slope equal to $F'(x_Q)$, or $f(x_Q)$. By repeating this process, using each new point to make a new segment, you can build an approximation for $F(x)$ one step at a time.

Q2 Suppose you had been doing this with paper and pencil, starting with point Q, and you wanted to draw the segment constructed in step 7. What tells you the slope of your segment? How could you have predicted, in the figure above, that the second segment would have been angled as it is shown?

Q3 For your function, predict what the third segment will look like and draw a sketch of it in the margin. Then check your answer by using the tool to create the third segment.

8. Adjust the slider for h so that $h \approx 0.25$. Then create at least ten more segments by clicking on the endpoint of the previous segment with the **Antiderivative Segment** tool.



Q4 Suppose you had been doing this with paper and pencil. How could you have predicted when a segment would have a positive slope? A negative slope? A slope of 0?

Q5 How do you know when to make a segment whose slope is greater than the previous segment? Less than the previous segment?

9. Go to page 2 of the document, and either print the page or make a rough sketch of the function on your own paper. Starting from point A, sketch the antiderivative of the function shown, either smoothly or as a series of segments.

10. Adjust the slider so that $h \approx 0.25$. Choose the **Antiderivative Segment** tool and construct enough segments, starting at point A, to cover the interval $[x_A, 2]$. Compare this with the sketch you made by hand.

Point Q's label is not showing because point Q is very close to point R.

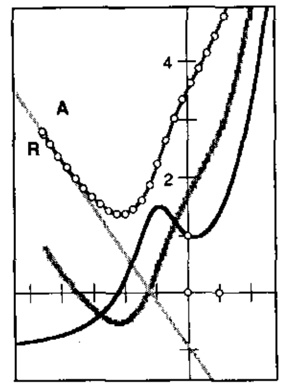
11. To compare your approximation with a very good approximation of the antiderivative, press the *Show RQ* button. Press the *Move R→A* button and then press the *Move R→Q* button.

Q6 How close was your hand-drawn sketch for the approximation of $F(x)$? How close was your approximation using the tool?

Q7 Adjust the slider for h so that $h \approx 0.1$. Describe what happens to the approximation you constructed with the tool and its accuracy. Explain why this happens.

You can also move point *A* up or down by pressing the Up or Down Arrow keys.

12. Adjust the slider for h so that $h \approx 0.25$ again. Select point A and drag point A up or down to a new location with the same x-coordinate. Press the *Move R→A* button and then the *Move R→Q* button to make a new trace at this new location.



Q8 What is the relationship between the new trace and the previous trace? Use your knowledge of derivatives to explain this relationship.

You can also use point *R* and the *Move* buttons to make a trace.

13. On printed copies for each of pages 3–6, sketch by hand the antiderivatives of the given function plots. Then, go to the sketch and use the tool to construct an approximation.

Q9 On page 6, the y-scale is not shown. Why is it still possible to sketch an approximate plot of the antiderivative of this function?

Use your work from pages 2–6 to answer the following questions.

Q10 When $f(x)$ is positive, what can you say about your antiderivative approximation? What can you say when $f(x)$ is negative?

Q11 When $f(x)$ is increasing, what can you say about your antiderivative approximation? What can you say when $f(x)$ is decreasing?

Q12 Where $f(x)$ changes from positive to 0 to negative, what can you say about your antiderivative approximation? What can you say about where $f(x)$ changes from negative to 0 to positive?

Explore More

When you constructed the tool on page 1 of this document, you selected the expression for $f(x)$ as one of the givens. So, if all you have is the plot of $f(x)$, this tool won't work. We need a different tool for that case.

1. Go to page 7 of the document. On this page, segments and a semicircle define $f(x)$. Try using your tool here. What happens?

Q1 When you sketch an antiderivative by hand, you don't really need the expression of $f(x)$—just the y-value. Sketch a graph of an antiderivative of the function $f(x)$ shown on page 7.

Q2 What happens at the points where $x = -2, 2, 4$, or 6?

In the steps below, you will make a new tool that addresses this case.

The point should be labeled *A* automatically. If not, use your **Text** tool to label this point *A*.

2. With the **Point** tool, construct a point on the semicircle. Measure the y-coordinate of this point by choosing **Measure | Ordinate (y).** Since

we lack an expression for $f(x)$, we can use this value, y_A, for the slope of the segment used in this tool.

3. Select the point again and measure its x-coordinate by choosing **Measure | Abscissa (x).**

4. With the **Point** tool, create an independent point, R, in the plane. Measure its x- and y-coordinates as you did above.

Compare this with the definition used in step 2 of Sketch and Investigate.

5. This time, define the tangent line function by choosing **Graph | New Function** and entering $y_R + y_A(x - x_R)$. Label this function $T(x)$.

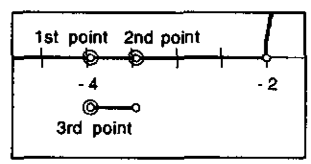
In place of the measurement h used earlier.

6. On a small interval, this linear function $T(x)$ will approximate the plot of the antiderivative. To define the length of this interval, create a second point anywhere on the semicircle. Measure the x-coordinate of this point, labeled C, and calculate $x_C - x_A$. Label this calculation h.

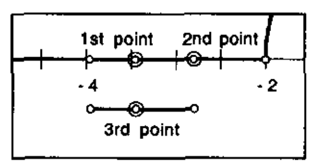
7. Choose **Measure | Calculate** and create the expression $x_R + h$. Then, calculate $T(x_R + h)$. Select $x_R + h$ and $T(x_R + h)$ and choose **Graph | Plot as (x, y)**. With the new point still selected, select point R and choose **Construct | Segment.**

8. Select points A, C, R, the new point constructed in step 7 (in that order), and the segment. Choose **Create New Tool** from **Custom** tools. Name it **Anti-Segment2.**

Your tool can now be used to construct an antiderivative of this and any other function with just the function's plot.



9. To use the tool, go to page 8. Choose **Anti-Segment2** from **Custom** tools. Construct two points on the first line segment of the function plot and then construct a third point directly below your first constructed point, as shown.



A segment with a slope equal to the y-value of the first point is created. The spacing between the two points on the function plot determines the segment's length.

10. Before you create more of the approximation of the antiderivative, make sure your first and third points have the same x-coordinate.

Q3 Why is it necessary that their x-coordinates are the same? Why did you not need to worry about this with the previous tool?

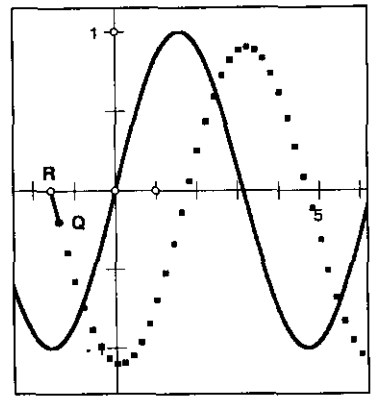
Make sure that your first two points are always on the function plot.

11. Use your tool to construct the antiderivative. So that your constructed antiderivative is all one piece, match your first and third points with the endpoints of the previous segment as shown above.

Q4 How close was your original sketch in Q1 to your constructed antiderivative?

Plotting the Antiderivative

Name(s): ____________________

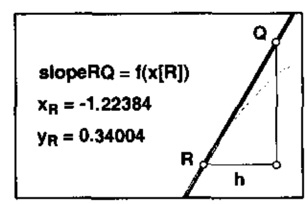


The process of constructing an approximate antiderivative, one linear step at a time, is called *Euler's method.* In the previous activity, it was very useful sketching antiderivatives, given the function's plot. In this activity, you will work with this process again—but this time with two different goals in mind. You will use Euler's method numerically to approximate the value of an antiderivative at any particular value of x and graphically to get an idea of the antiderivative expression.

Sketch and Investigate

1. **Open** the document **Antiderivatives.gsp** in the **Exploring Antiderivatives** folder. On page 1 of this document you will find a function $f(x)$, its plot, a fixed point R, and a parameter h. Here $f(x)$ is the derivative of some unknown function F, so $F'(x) = f(x)$.

Q1 Suppose you are given initial values x_R and y_R of a point on $F(x)$. Using $f(x_R)$ as the slope of the plot of $F(x)$ at point R, what is an approximate value of $F(x)$ at $x_R + h$, shown in the figure as point Q?



Euler's method is an iterative algorithm: start with an x- and y-coordinate, perform the algorithm, and then repeat on the new coordinates. In Sketchpad, you can actually see this process using the **Iterate** command.

2. Here, point R has coordinates determined by the parameters x_R and y_R. Select the measurements x_R and y_R, and then choose **Iterate** from the Transform menu. Click on $x_R + h$ and then $y_R + f(x_R) \cdot h$ to match the pre-images to their first images. Uncheck **Tabulate Iterated Values** in the Structure pop-up menu. Press the *Iterate* button.

You can select the iterated image by clicking on any of the three image points with the **Arrow** tool.

3. The first three iterations of Euler's method will appear in your sketch. With the iterated image selected, choose **Properties** from the Edit menu. Change the number of iterations to 100 on the Iteration panel.

You can check your answer by plotting your prediction as a new function.

Q2 Using just the iterated plot given by Euler's method in step 3, can you predict what the antiderivative might be?

Next, we'll use Euler's method to go step by step to approximate $F(2)$ given that $f(x) = \sin(x)$ and we start at $(-2, 1)$.

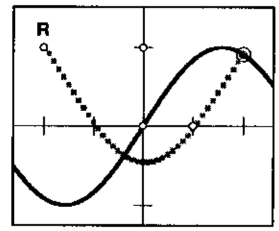
This requires us to figure out how many steps there are of size h from the initial point to the terminal point. In this example, if the initial point has

$x_R = -2$, final point at $x = 2$, with step size 0.1, then $(2 - (-2))/0.1 = 40$, so we need 40 iterations.

4. Select the iterated image by clicking on any of the points in the image; then choose **Properties** from the Edit menu and change the number of iterations to 40 on the Iteration panel.

You can select an iterated image by clicking on any of the image points with the **Arrow** tool.

5. To construct the terminal point, choose **Terminal Point** from the Transform menu with the iterated image selected.



6. With the terminal point still selected, measure its coordinates by choosing **Measure | Coordinates**.

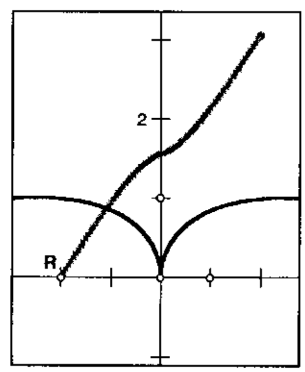
Q3 Is the terminal x-value after 40 steps of size 0.1 indeed 2? If so, what is the approximate value for $F(2)$ using Euler's method with a step size of 0.1?

To change a parameter's value, double-click on it with the **Arrow** tool and enter a new value.

Q4 What is the approximate value for $F(2)$ using Euler's method with a step size of 0.05? (Change parameter h and the number of iterations.)

Q5 Describe how the iterated image changed with this new step size.

You might already know the antiderivative of $\sin(x)$, so you could calculate $F(2)$ directly (always remembering the initial conditions), but in many cases it is not possible to find an expression for the antiderivative of a function, so your only option is Euler's method. Here's an example.



7. Double-click on the expression for $f(x)$ and change it to `f(x)=sin(sqrt(abs(x)))`.

You will need to change the value of the parameter y_R here.

Q6 Now, you can't find an expression for $F(x)$, but, using Euler's method with a step size of 0.05, what is the approximate value for $F(2)$, if $F(-2) = 0$?

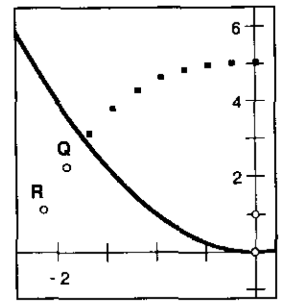
You can use this method with any function, whether it has an explicit antiderivative or not. For the ones that do, sometimes it takes a lot of practice to antidifferentiate correctly. Here, Euler's method can give you a rough plot to start with.

8. Go to page 2 of the document. Here we have the same situation as on page 1, except that here point R is independent or free to move.

9. Select the measurements $x_R + h$ and $y_R + f(x_R) \cdot h$ in that order and choose **Plot as (x, y)** from the Graph menu. Label this point Q. With this new point, you can repeat Euler's method.

Make sure point Q is not selected as well by clicking once in an empty spot with your **Arrow** tool.

10. Select point R, and choose **Iterate** from the Transform menu. Click on point Q to match the pre-image (point R) with the first image (point Q). Uncheck **Tabulate Iterated Values** in the Structure pop-up menu. Click the *Iterate* button, and three iterations will be constructed.



11. With the new iterated image selected, choose **Edit | Properties.** Change the number of iterations to 200 on the Iteration panel.

12. Use the slider to adjust the value of h to approximately 0.02.

You can enter your guess using the Graph menu and then see how well it matches the iterated image.

Q7 Do you recognize this plot? Can you guess its equation?

Exploration 1

On pages 3–7 there are various functions and their plots. For each one, an iterated image using Euler's method with step size h has already been constructed.

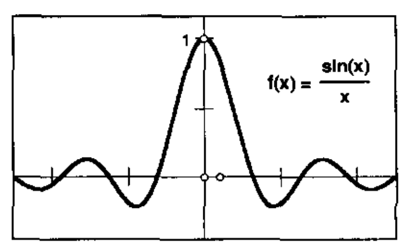
1. For each function $f(x)$, use the iterated image to predict the expression for the antiderivative. Write your prediction on a separate sheet of paper.

2. Test your prediction by choosing **Plot New Function** from the Graph menu and entering your expression. If you need some constants, press the *Show Sliders* button and use the sliders provided to build the expression.

3. If your prediction is not correct, compare the plot of your guess and the iterated image for hints as to how you should edit your function.

4. As you saw in step 7, there are functions for which it is not possible to find an antiderivative expression using elementary functions. Examine the function f on page 8. Iterate point R as you did before, and examine the image created.

Q1 Sketch your iterated image of the antiderivative that you constructed in step 4 on your paper. What is the approximate value for $F(5)$ using Euler's method with a step size of 0.05, if $F(-10) = -1.75$?



Exploration 2

You have seen that when a function is translated vertically, its derivative is unchanged. Is its antiderivative unchanged as well?

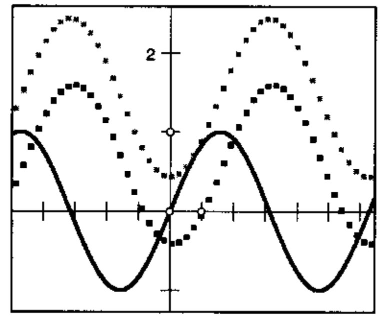
1. Go to page 9 of the document. On this page, $f(x) = \sin(x) + c$.

Q1 Adjust the slider for c. What happens to the antiderivative approximation, or the iterated image, when you translate the plot of $f(x)$ vertically upward? Vertically downward? Why does this happen?

Q2 In general, how will the antiderivative of $f(x)$ differ from the antiderivative of $f(x) + c$?

The sine function, its plot, and its antiderivative approximation are also useful for looking at other transformations.

2. Go to page 10 of the document. Here you'll find the function $f(x) = \sin(x)$, a transformation of $f(x)$ labeled $g(x)$, a couple of sliders, and the antiderivative approximations for each function constructed as before.



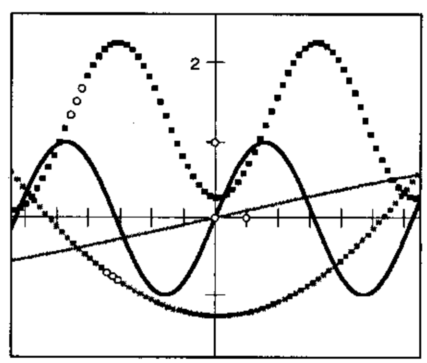
3. Adjust the slider for c and investigate how changing the value of c affects the approximated antiderivative plot of the function $g(x)$, or $f(x + c)$.

Q3 Are the antiderivatives of $f(x)$ and $f(x + c)$ horizontal translations of each other for all initial conditions? If not, can you adjust the initial conditions so that this is the case? Or is there some other relationship between the antiderivatives? If so, what is it?

4. Edit the function g so that $g(x) = c \cdot f(x)$. Investigate the relationships as you did in step 3.

Q4 Are the antiderivatives of $f(x)$ and $c \cdot f(x)$ vertical stretches of each other for all initial conditions? If not, can you adjust the initial conditions so that this is the case? Or is there some other relationship between the antiderivatives? If so, what is it?

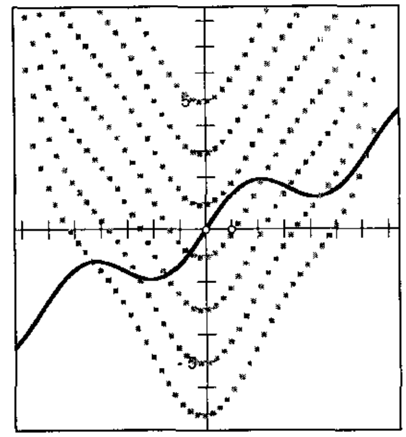
5. Edit the function g so that $g(x) = f(c \cdot x)$. Investigate the relationships as you did in step 3.



Q5 Are the antiderivatives of $f(x)$ and $f(c \cdot x)$ horizontal compressions of stretches of each other for all initial conditions? If not, can you adjust the initial conditions so that this is the case? Or is there some other relationship between the antiderivatives? If so, what is it?

A Field of Slopes

Name(s): ____________________



Finding the antiderivative $F(x)$ from the equation of $f(x)$ where $F'(x) = f(x)$ is called solving a *differential equation.* As you have seen, a differential equation has many solutions, because if $F(x)$ is an antiderivative of $f(x)$, then so is $F(x) + C$ for any constant C. So every differential equation actually describes a family of functions that are translations of each other.

In this activity, you will investigate *slope fields,* which are a method for visualizing the family of functions described by a differential equation, and apply a method for approximating a particular solution from the many possible solutions.

Sketch and Investigate

1. **Open** the document **Slopefield.gsp** in the **Exploring Antiderivatives** folder. On page 1 of this document you will find a function $f(x)$, its plot, an independent point R, and a slider.

Select point *R*, then choose **Measure | Abscissa (x).** (Choose **Ordinate (y)** for the *y*-coordinate.)

2. Because point R is independent, we can assume that it is on the antiderivative F. Measure the x- and y-coordinates of point R.

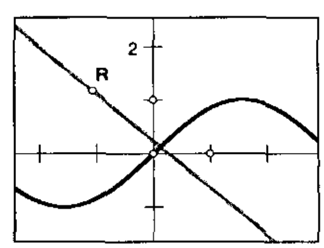
In the previous activities you learned that if $F(x)$ is a solution to the differential equation $F'(x) = f(x)$ and $F(x)$ contains point R, then at point R, the tangent line to F must have the equation $y_R + f(x_R)(x - x_R)$ and near point R, the antiderivative $F(x)$ must *look like* this tangent line.

3. Calculate $f(x_R)$ by choosing **Measure | Calculate.** Label this calculation *SlopeAtR* by double-clicking on it with your **Text** tool.

Click on the measurements to enter them into the Function editor.

4. To plot the tangent line, choose **Plot New Function** from the Graph menu and enter the expression $y_R + SlopeAtR \cdot (x - x_R)$.

5. Relabel this function T by double-clicking on the equation with your **Text** tool.



The tangent line only approximates the plot of $F(x)$ for a small interval around point R, so next you'll create a short segment of a fixed length around point R, with a slope equal to the slope of the tangent line by using the slider that adjusts the measurement s.

Click on the measurements to enter them into the calculator.

6. Choose **Measure | Calculate** and calculate $x_R + s$. Then calculate $T(x_R + s)$.

7. Select $x_R + s$ and $T(x_R + s)$ in that order, then choose **Plot As (x, y)** from the Graph menu. Label this point Q.

8. Plot the point $(x_R + s, y_R)$.

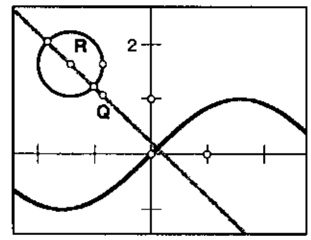
Or, using the **Circle** tool, you can click on point R, and then on the new point.

9. Construct a circle with center R and radius s by selecting point R and the new point from step 8 and choosing **Circle by Center+Point** from the Construct menu.

10. Select the tangent line and choose **Display | Hide Function Plot.** Then select points R and Q and choose **Construct | Line.**

You can also make a segment by selecting both points with the **Segment** tool.

11. Construct both intersections of the circle and the line RQ by clicking on the intersections with your **Arrow** tool. Then construct a segment between these two intersections by clicking on both points and choosing **Construct | Segment.**

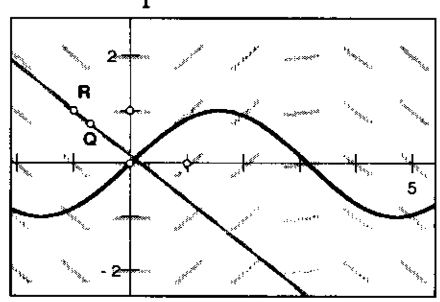


12. Hide the point $(x_R + s, y_R)$, the circle, and both intersection points by selecting them all and choosing **Display | Hide Objects.**

13. Adjust the slider for s to approximately 0.3.

The segment you've created centered at point R has slope $F'(x_R)$ or $f(x_R)$ and a fixed length. Now for the fun part! The segment at R shows you the slope at a single point. Dragging R shows the slope at *every* point of the drag. A *slope field* diagram describes the slope at many points simultaneously, giving a "big picture" of the shape of all solutions to the differential equation.

14. Select the segment and choose **Display | Trace Segment.** Make the segment thicker by choosing **Display | Line Width | Thick.** Using tracing here shows the segment's slope at various positions of R simultaneously.



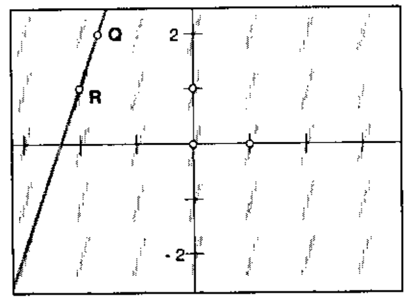
15. Choose **Snap Points** from the Graph menu. Snap points causes dragged points to stick to the grid points of your coordinate system.

16. Drag point R. As you drag, the traced segment shows the local slope for each point on the grid. Try to drag R so that you fill in the entire grid.

Q1 In your slope field, where is the segment's trace horizontal or closest to horizontal? Where is the trace closest to vertical?

Q2 Explain your answer to the previous question algebraically in terms of your differential equation. Why is the line horizontal or near-vertical in the cases you found in Q1?

Q3 Suppose you dragged point R around the plane and traced the slope field in the picture on the right. What can you say about the differential equation?

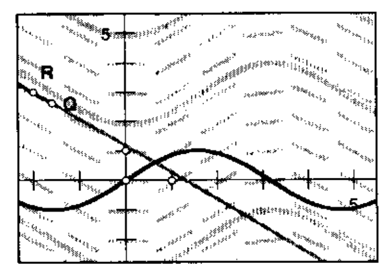


Now, in order to find a particular solution to a differential equation, you must have at least one point that lies on $F(x)$—called an *initial point*. Suppose, for example, that $F(x)$ passes through the point $(-9, 7)$. You can now trace an approximate particular solution by "following" the tangent line.

17. Select point R, and then point Q, and choose **Edit | Action Buttons | Movement.** Click OK. This button will move point R toward point Q so that point R will *follow* the tangent line.

18. Move point R to the point $(-9, 7)$. Then press the *Move R→Q* button.

Q4 Given that the antiderivative of $\sin(x)$ is $-\cos(x) + C$, find the actual equation of the function that you've approximated by this trace. Plot your answer to see how it compares with the approximation.



If point R moved off your screen, choose **Edit | Undo Animate Point** to bring it back to its starting point.

Q5 Move point R to another location and press the *Move R→Q* button. Find the equation for this new trace as you did in Q4. Is this trace a vertical translation of your answer in Q4?

Looking at the traces of the approximate particular solutions, you can see that the segment traces that make up the slope field suggest the slope or direction of all the possible solutions to a given differential equation.

Q6 Go to page 2 of the document. Here $f(x) = x$. Predict what the slope field will look like for this differential equation, and then create the field as you did in step 16 above.

If you don't have access to a printer, copy the diagram on page 109.

Q7 Once you've diagrammed the slope field for your differential equation, choose **File | Print** to print your sketch. Describe the shape and location of the patterns that occur in the slope field.

Q8 On your printout, use different colored pencils to trace out three particular solutions you see in your slope field.

In the questions above, you could probably predict what the slope field and the particular solution would look like from what you know about derivatives and antiderivatives. But slope fields can also be used for differential equations whose solutions aren't so obvious.

The alternative symbol for derivative is used here because the differential equation is a function of both x and y.

For instance, what function satisfies the differential equation $\frac{dy}{dx} = y$?

What does its slope field look like? Here the slope of the solution at a point depends on y instead of x. In other words, at any point (x, y), the solution has a tangent line with slope equal to the y-coordinate of that point. So, for example, at the point (2, 3), the solution $F(x)$ has a tangent line with slope 3. Is there a function with this property? You can use slope fields to find out.

The grid spacing can be adjusted by pressing the *Show Grid Controls* button and then editing the parameter *scale*.

19. Go to page 3 of the document. On this page you will find a point R and a segment centered at R, as on page 1. Here, though, point R is constrained to move along the locus of grid points so that you can create a slope field with more detail.

Q9 Use a piece of graph paper to sketch a slope field for this differential equation. At each point R, (x_R, y_R), draw a small segment whose slope is equal to the value of y_R.

20. To check your answer for Q9, select *SlopeAtR* and choose **Edit | Edit Calculation.** Then enter in the expression y_R for the new slope.

21. To create the slope field, move point R to every grid point.

If you don't have access to a printer, copy the diagram on page 109.

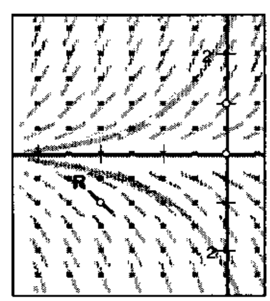
Q10 Print your sketch by choosing **File | Print.** Describe the shape and location of the patterns that occur in the slope field.

Q11 On your printout, use different colored pencils to trace out three different particular solutions you see in your slope field.

22. To check your answer to Q11, press the *Show Initial Pt* button and move point *Initial* to any one of your initial conditions in Q11.

23. Select point R and choose **Edit | Split Point from Locus** and press the *R→Initial Pt* button. Now press the *Move R→Q* button.

Q12 What function does your particular solution look like? Using your knowledge of derivatives, can you prove that it is this function?



Explore More

For each differential equation answer these questions.

A. $\frac{dy}{dx} = 0.5 \cdot x$ B. $\frac{dy}{dx} = y - 2$

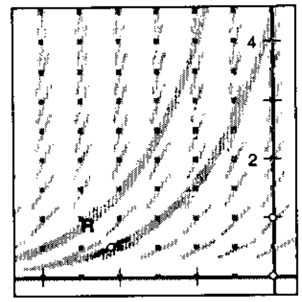
Q1 Describe how the slope field is similar to and different from $\frac{dy}{dx} = y$.

Select point R and any point in the grid, then choose **Edit | Merge Point To Locus.**

Q2 Repeat steps 19–23 and answer Q9–12 for each equation.

Stepping Through the Field

Name(s): ____________________



In the previous activity, you were introduced to slope fields as a way to visualize the family of solutions to a differential equation. You saw that slope fields could be used as a map—pick a starting point and follow the direction of the segments through the field.

Slope fields *point the way* along solution curves by showing the slope of the tangent to the solution curve. In this activity, you will see how Euler's method tells you *how* to move through a slope field numerically, step by step. You will also explore a wider variety of slope fields, use them as maps to draw possible solution curves, and apply Euler's method to approximate a particular solution numerically.

Sketch and Investigate

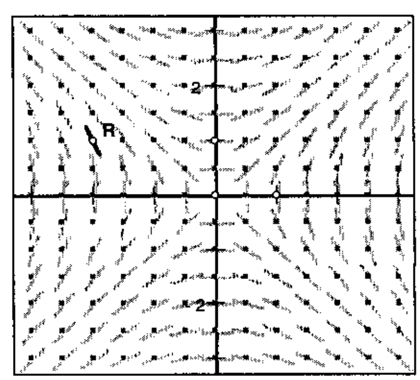
To start the process, let's examine the differential equation $\frac{dy}{dx} = \frac{x}{y}$.

What does its slope field look like? Here the slope of the solution at a point depends on both x and y, or in other words, at any point (x, y), the solution has a tangent line with slope x/y. So, for example, at the point (2, 3), the solution $F(x)$ has a tangent line with slope 2/3.

Q1 Use a piece of graph paper to sketch a slope field for this differential equation. At each point, $\{(x_R, y_R) \mid -3 \le x_R \le 3, -3 \le y_R \le 3\}$, draw a small segment with slope x_R/y_R.

The grid is really a locus of grid points, and point *R* is on the locus.

1. **Open** the document **Euler.gsp** in the **Exploring Antiderivatives** folder to check your answer. On page 1 of this document there is a grid of points. Point *R* is on the grid, and the segment centered at point *R* has been constructed so that its slope is equal to x_R/y_R.



2. Check your slope field by dragging point *R* around the grid—make sure you have a trace at all the grid points.

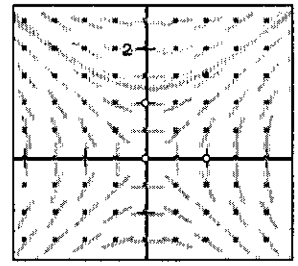
If you don't have access to a printer, copy the diagram on page 109.

Q2 Choose **File | Print** to print your sketch. Describe the shape and location of the patterns that occur in the slope field.

Q3 On your printout, use different colored pencils to trace out three different particular solutions in your slope field.

3. To test one of your particular solutions, press the *Show Initial Pt* button and move point *Initial* to one of your initial points in Q3.

4. Select point R and choose **Edit | Split Point from Locus**. Press the *R→Initial Pt* button and then the *Show Point Q* button.



Adjust the slider labeled *Direction* to move point R in the opposite direction.

5. Now, press the *Move R→Q* button. Observe how point R follows the direction suggested by the segments and the slope field.

In "Plotting the Antiderivative," you saw that Euler's method is an iterative algorithm with these steps:

i. Start with an initial point or condition or (x_R, y_R).

ii. Calculate the slope at that point using the given equation—in this case, x_R/y_R.

iii. For the given value of h, calculate the change in y using this slope —specifically, $slope \cdot h$. Add this calculation to y_R to get a new point, $(x_R + h, y_R + slope \cdot h)$.

iv. Repeat this process with the new point.

Now, let's apply Euler's method using (–5, 4) as the initial point.

To remove clutter, you can hide point *Initial*'s label by clicking on it once with the **Text** tool.

6. Select point *Initial* and any grid point to select the locus of grid points. Choose **Edit | Merge Point To Locus.** Now move point *Initial* to the point (–5, 4). Press the *R→Initial Pt* button. The slope of the solution at this point is x_R/y_R, or –5/4.

Because h is set at 0.2, the y-value of the first approximation using Euler's method will be $y_R + slope \cdot h = 4 + (-1.25)(0.2) = 3.75$. So the first approximated point will be $(x_R + h, y_R + slope \cdot h)$, or (–4.8, 3.75).

7. Point Q has been constructed using Euler's method. Choose **Measure | Coordinates** to measure point Q's coordinates. Do the coordinates agree with the calculations above?

Check the status line to make sure that you have selected point R. If not, click on point R again.

8. Press the *Hide Segment* button. To see Euler's method repeated, select point R and then choose **Iterate** from the Transform menu. Click on point Q to match the image point. Set the number of iterations to 1 by pressing the minus (–) key twice. Uncheck **Tabulate Iterated Values** in the Structure pop-up menu. Click Iterate.

Q4 By hand, apply Euler's method to the point (–4.8, 3.75). What is the next approximated point?

The choice, **Terminal Point**, will only show up if you have selected only the last created point.

9. To check your answer to Q4, select the iterated image by clicking on the last created point, and choose **Transform | Terminal Point.**

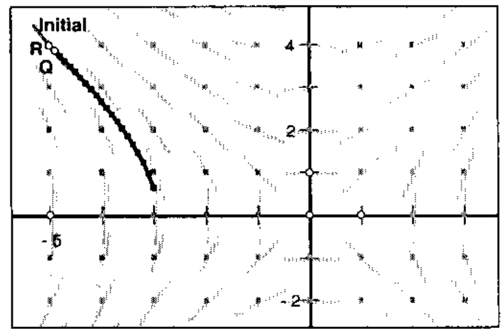
10. With the terminal point selected, choose **Measure | Coordinates** to measure its coordinates. Do the coordinates agree with Q4?

Make sure the status line says "Selected: 1 iterated image". Otherwise, click on the point again.

11. Click on the iterated image and increase the number of iterations by pressing the plus (+) key.

Q5 What happens when you use Euler's method with an h of 0.2 after a total of 12 iterations (counting point Q as the first iteration)? Explain why this happens.

12. Select parameter h, choose **Edit | Edit Parameter**, and change its value to 0.1. Then, again select the iterated image, but this time, choose **Edit | Properties**. Change Number of Iterations to 19 on the Iteration panel. Click OK.



Q6 Your terminal point has coordinates (–3, 0.643). What do these coordinates mean in relation to the differential equation?

When you click on the location of the terminal point, the iterated image will be selected first.

Q7 How could you use your sketch to approximate $F(3)$ using Euler's method with initial point (1, 0.5) and step size $h = 0.1$? Try it.

You can use this document to experiment with all sorts of different differential equations by following the steps below.

13. Go to page 2 to experiment with differential equations that have a variable in the denominator. Examine the equation $\frac{dy}{dx} = \frac{x_R + y_R}{y_R^2 + 1}$ by editing the calculation *numerator* to equal $x_R + y_R$, and *denominator* to equal $y_R^2 + 1$. (The calculation *SlopeAtR* will be updated automatically.) Move point R around the grid to create a trace of the slope field.

14. Go to page 3 to experiment with differential equations that do not have a variable in the denominator. Examine the equation $dy/dx = (x_R - y_R)/3$ by editing the calculation *SlopeAtR* directly. Move point R around the grid to create a trace of the slope field.

Exploration 1

Pages 4 and 5 of the document have slope fields already constructed. These are hidden when you open the page so that you can trace the slope field yourself as you did above. Pick a couple of the differential equations below and use pages 4 and 5 to do the following steps. (Use page 4 to explore to explore differential equations with constant denominators and page 5 for differential equations with variable denominators.)

A. $dy/dx = ay^2$ B. $dy/dx = ay^2/x$

C. $dy/dx = a(y/x)$ D. $dy/dx = 0.5x + y$

E. $dy/dx = ax(1 - x)$ F. $dy/dx = ax/by$

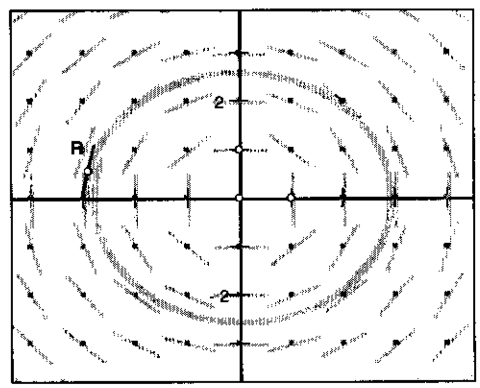
G. $dy/dx = cx/(y^2 + x^2)$ H. $dy/dx = cy/(y^2 + x^2)$

Q1 For each, first predict what the slope field will look like by evaluating the differential equation at a few different points (x, y).

1. Then, edit the expression for the slope as described above in step 13 or 14 and make your slope field. Press the *Show Field* button, then use the sliders to vary the values of the parameters in the differential equations and observe any changes in the slope field.

If you don't have access to a printer, copy the diagrams on pages 109 and 110.

Q2 Print your sketch by choosing **File | Print.** Describe the shape and location of the patterns in the slope field.



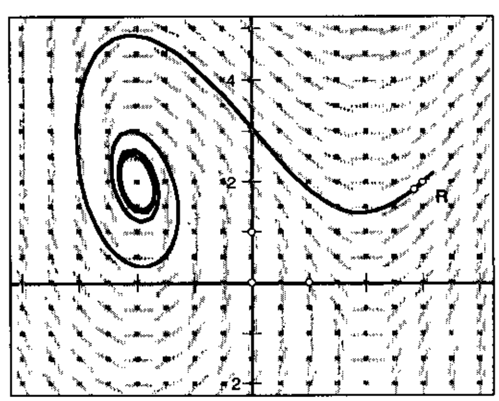
Q3 On your printout, use different colored pencils to trace out three different particular solutions in your slope field.

2. Create a particular solution by using the **Iterate** command as described above in step 8.

Q4 Describe how your particular solutions are similar and how they are different for different initial conditions.

Exploration 2

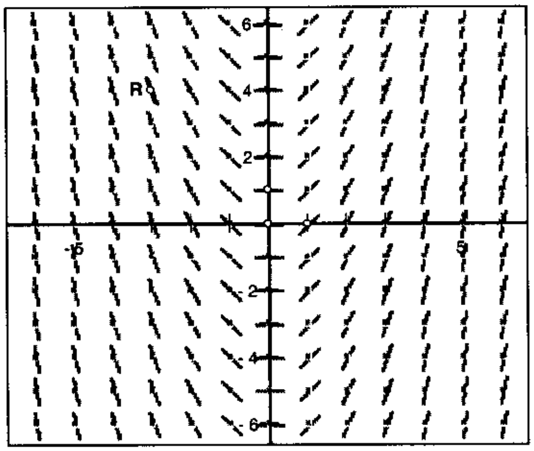
In Q5 of the Sketch and Investigate section, you saw that Euler's method can lead to poor approximations when vertical tangents are involved. Other methods for approximating solutions to differential equations are more precise.



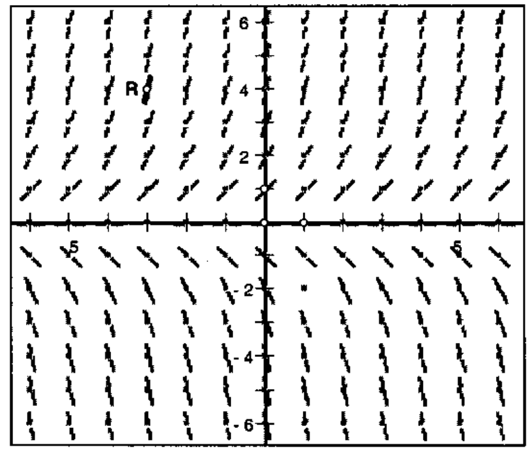
Q1 What is it in the differential equation on page 6 that causes the "whirlpool" effect in this solution curve?

A Field of Slopes:

Q7 $f(x) = x$

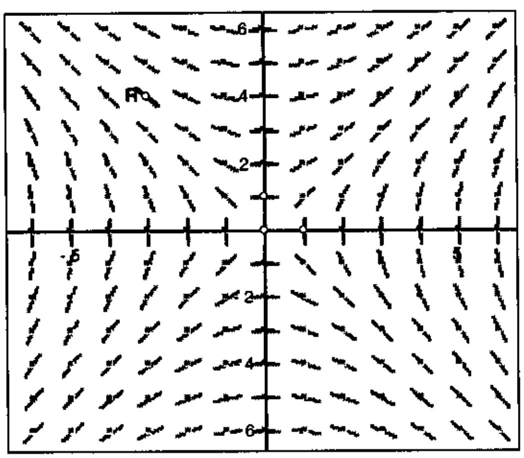


Q10 $dy/dx = y$

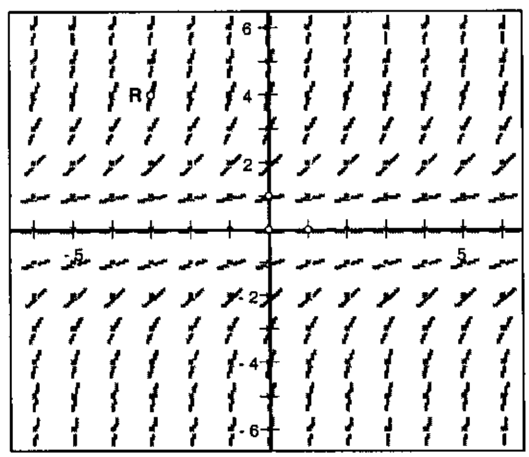


Stepping Through the Field:

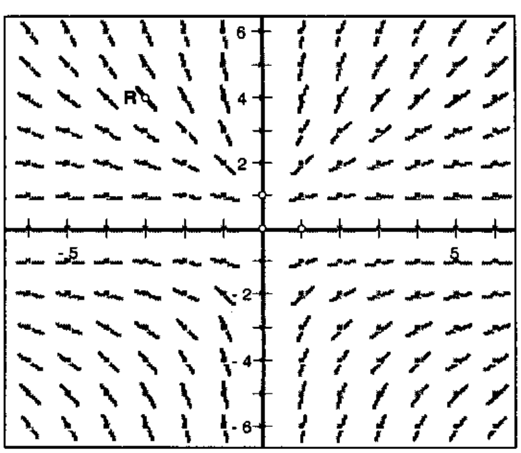
Q2 $dy/dx = x/y$



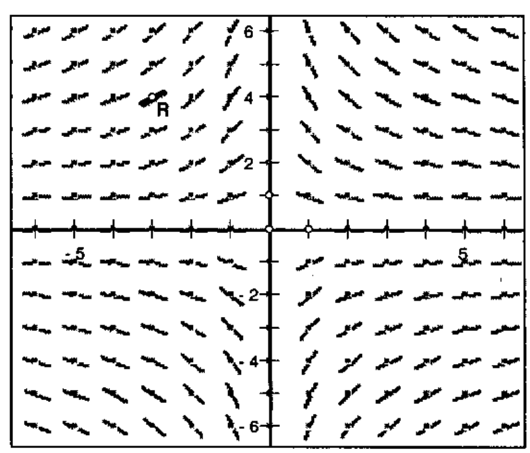
A. $dy/dx = ay^2$



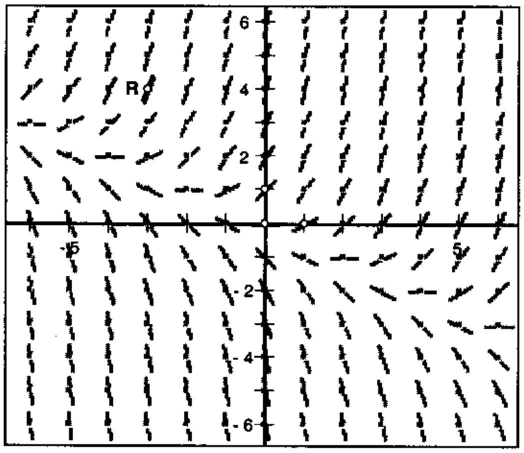
B. $dy/dx = ay^2/x$



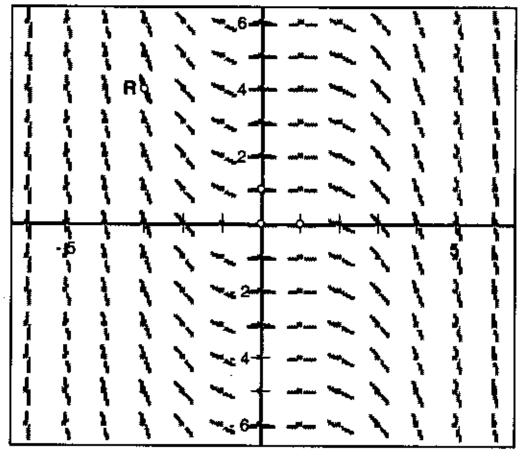
C. $dy/dx = a(y/x)$



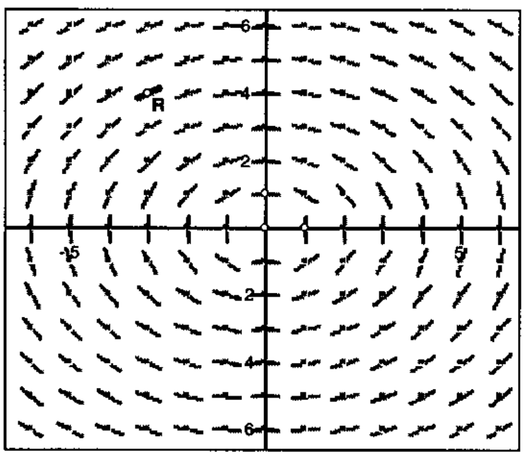
D. $dy/dx = 0.5x + y$



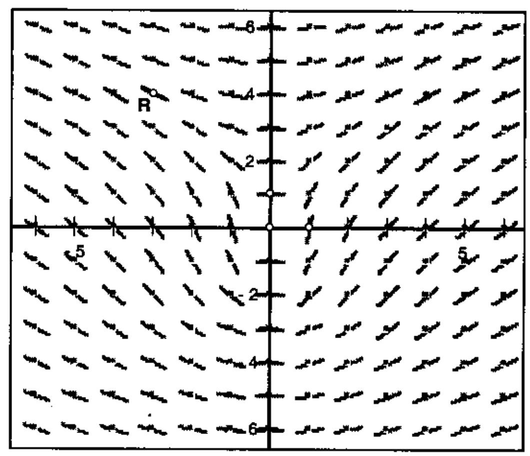
E. $dy/dx = ax(1 - x)$



F. $dy/dx = ax/by$



G. $dy/dx = cx/(y^2 + x^2)$



H. $dy/dx = cy/(y^2 + x^2)$