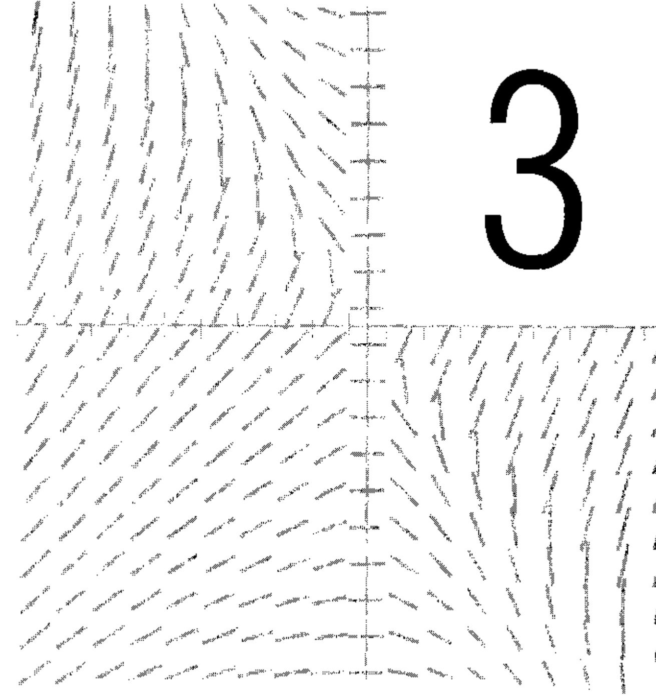
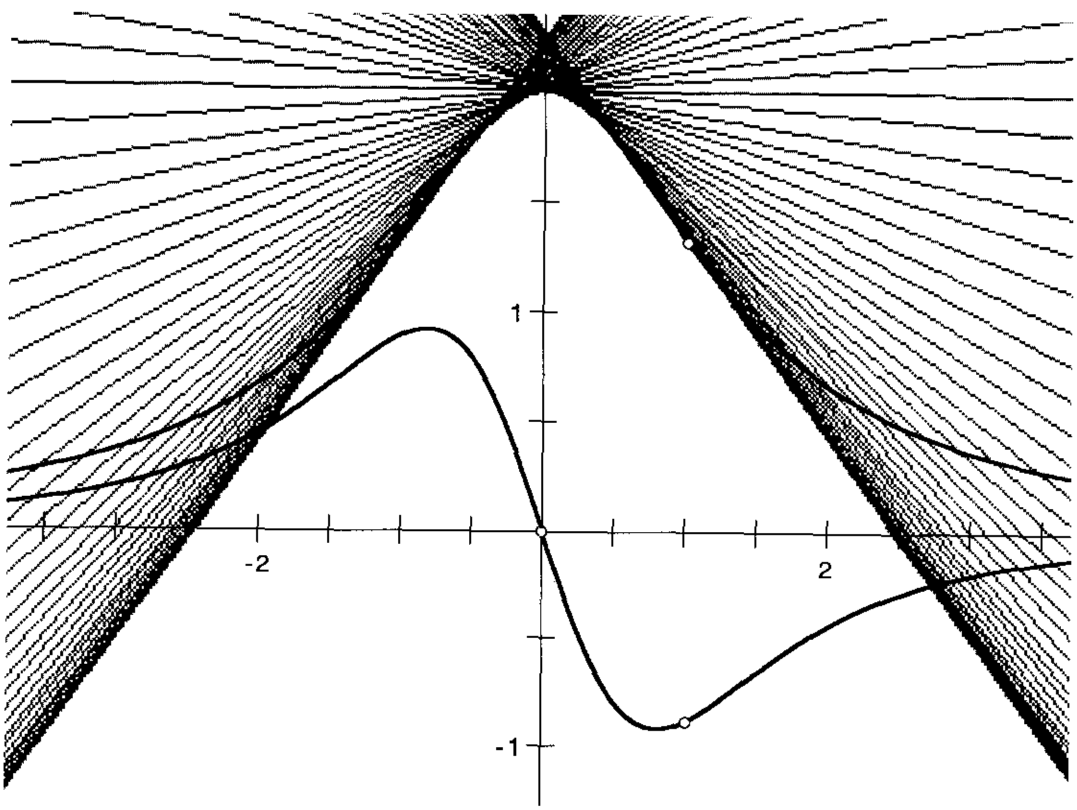


3

Exploring Derivatives



Taking It Near the Limit

Name(s): ____________________

If you want to model something in the real world with a function, like the gallons of gas your car uses per mile, it would be nice if the conditions always gave the same result—in this case, your mileage alone determines the number of gallons used. Unfortunately, the rate at which your car uses gas changes for all sorts of reasons—how fast you are going, your terrain, or even when you last changed your oil. A function is a good model, but we need something more. We need something that can predict and describe changing quantities as well as static ones.

The derivative of a function is such a tool. For a function $y = f(x)$, the *derivative* at a point is the instantaneous rate of change of f with respect to x at that point. In other words, the derivative tells you how much y is changing at any particular value of x. How do you find this derivative, and what does it look like? This activity will help you find out.

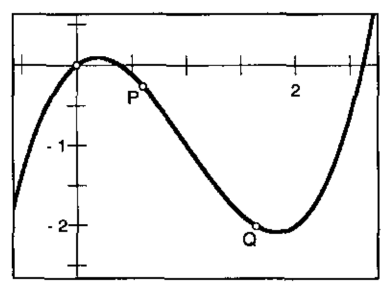
Sketch and Investigate

In order to start this process, you need a function.

1. **Open** the sketch **Derivative1.gsp** in the **Exploring Derivatives** folder. Choose **Plot New Function** from the Graph menu and enter $x^3 - 3x^2 + x$. Label this function f.

To construct a point, choose the **Point** tool and click on the function plot. Then with the **Text** tool, double-click on the point and type in your new label.

2. Construct two points anywhere on the function plot. Label these points P and Q.



3. Select point P and point Q and measure their x-coordinates by choosing **Abscissa(x)** from the Measure menu.
4. Calculate $f(x_P)$ and $f(x_Q)$. Choose **Measure | Calculate.** Select the expression for f (not the plot!) to enter it into the calculator, and then select the measurement x_P. Repeat for point Q.

Recall that the average rate of change from $P\ (x_1, y_1)$ to $Q\ (x_2, y_2)$ is represented graphically as the slope of the secant line through P and Q:

$$rate_{avg} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = slope(PQ)$$

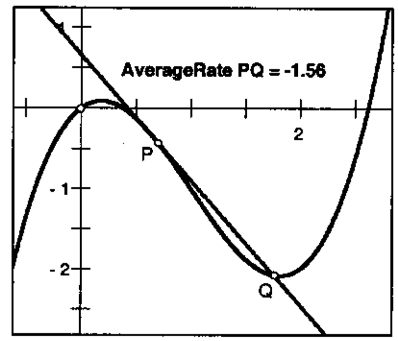
Q1 In terms of the measurements you took in steps 3 and 4, what is the slope of the line PQ?

5. Construct the secant line PQ by selecting point P and point Q and choosing **Line** from the Construct menu.

Always select the measurement to enter into the calculator—not the point!

6. Find the average rate of change between the two points by calculating the slope of the line PQ. Choose **Measure | Calculate,** and then click

on each measurement to enter it into the calculator. Label this measurement *AverageRate PQ.*



Your average rate of change (or secant slope) for *f* between point *P* and point *Q* tells you how much the function's *y*-value changes on the interval from x_P to x_Q. An instantaneous rate tells you how much the *y*-value is changing at a point—in this case, at point *P*. You can't use the slope formula with only one point (why not?) but you can look at the limit of the average rate or secant slope as point *Q* approaches point *P*.

7. Move point *P* so that its *x*-coordinate is as close to $x = 2$ as possible. You may not be able to get to $x_P = 2$ exactly, but that's okay.
8. Move point *Q* so that its *x*-coordinate is as close to $x = 2.1$ as possible.
9. Write down the *x*-value of 2.1 and the value of *AverageRate PQ* on your paper.
10. Move point *Q* so that its *x*-coordinate is as close to $x = 2.05$ as possible.
11. Write down the *x*-value of 2.05 and the value of *AverageRate PQ*.

Q2 From these two readings, what estimate would you give for the right-hand limit? If you place point *Q* as close as possible to $x = 1.9$ and $x = 1.95$, does the left-hand limit agree with the right-hand limit?

You may not be able to line them up exactly—if you can't, what *should* happen?

Q3 What happens to the slope measurement and the line if you actually put point *Q* in the same spot as point *P*? (After you do this, move point *Q* away from point *P* so they are distinct points again.)

You can zoom in at point *P* to get a better estimate for the limit or to check your estimate.

You can change *a* or *b* by double-clicking on the measurement and then typing in a new value.

12. Press the *Show Zoom Tools* button. The values *a* and *b* are the coordinates of the point you can zoom in on. The point *P* has coordinates (2, –2), so *a* should be set to 2 and *b* to –2.
13. Adjust the *x*- and *y*-scales at the same time by selecting points *x-scale* and *y-scale,* then click on either point and drag. Now when you move your mouse both sliders will move together. Try it.

As you zoom in, the axes should disappear. Press the *Show Side Axes* button. You can move these axes by dragging them.

14. Zoom in until your *x*-side-axis goes from approximately 1.2 to 2.4.

Although point *Q* will appear to move away from point *P*, note that the coordinates of point *Q* do not change. If you can, move point *P* closer to $x = 2$.

15. Move point *Q* so that its *x*-coordinate is as close to $x = 2$ as possible.

16. Write down the x-coordinate of point Q and the value of *AverageRate PQ* on your paper.

Q4 What estimate would you now give for the limit of the average rate or secant slope? (Do both sides still agree?)

If the tick marks on your axes disappear, zoom back out a little.

17. Zoom in one more time, but this time keep zooming in until your curve looks like a straight line. Again move point P as close to $x = 2$ as possible. (You should be able to be right at $x = 2.00000$.)

18. Move point Q so that its x-coordinate is as close to $x = 2$ as possible.

Q5 What estimate would you now give for the limit of the average rate or secant slope? Do both sides agree? Did it change much from your first initial estimate?

Q6 Around point P, can you tell the difference between a point on the line and a point on the curve?

You should have gotten a value of 1 for your limit of the average rate at $x = 2$.

$$\lim_{Q \to P} \textit{average rate from } P \textit{ to } Q = \textit{instantaneous rate at } P = \textit{the derivative at } P$$

so we write

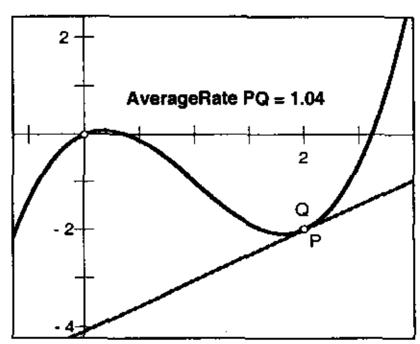
$$\lim_{Q \to P} rate_{avg} = \lim_{Q \to P} \frac{y_Q - y_P}{x_Q - x_P} = \lim_{Q \to P} \frac{f(x_Q) - f(x_P)}{x_Q - x_P} = f'(x_P)$$

or simply $f'(2) = 1$. The notation f' stands for the instantaneous rate, or derivative.

Q7 You know that the graphical representation of the average rate of change from point P to point Q is the slope of the secant line PQ. Above, we've defined that

$$\lim_{Q \to P} \textit{average rate from } P \textit{ to } Q = \textit{instantaneous rate at } P$$

What do you think would be the graphical representation of the instantaneous rate (or derivative)? *Hint:* Zoom out (select both sliders and drag left) and look at the relationship between the function plot and the secant line. (Do not move point P or point Q.) What kind of line does the secant line look like now? We'll come back to Q7 and this idea in the next activity.



Now that you're zoomed out, you're ready to look for another instantaneous rate or derivative. An interesting place to look is at a point where the function changes direction.

19. Move point P as close as you can to the *local maximum point*—basically the top of a hill where f changes from going up to going down.

Before zooming in, be sure to change the value of a to the x-coordinate and b to the y-coordinate of the point where you want to zoom.

20. Move point Q as close as you can to point P. (See side note.) Zoom in a few times to get a good approximation for the x-coordinates of both the maximum and the limit.

Q8 What value for x_P gave you the biggest y-value or maximum? How can you be sure that your value for x_P is the best choice? What maximum value did you get?

Q9 What value did you get for the limit of the average rate for this point P?

Q10 For this point P, what is $\lim\limits_{Q \to P} \dfrac{f(x_Q) - f(x_P)}{x_Q - x_P}$, or $f'(x_P)$?

Q11 Use the sliders to zoom back out (but don't move point P or point Q). Looking at your secant line, what would you say about its slope?

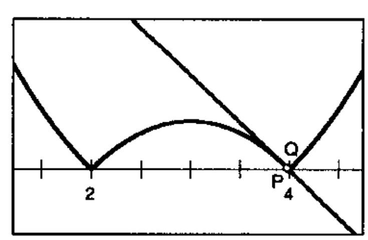
Explore More

For each of these new functions and values, don't forget to change a and b to the center of where you want to zoom in!

1. Zoom out until your x-axis goes from approximately $x = -2$ to $x = 6$. Double-click on the expression for $f(x)$ and change the function to $f(x) = \sin(x)$. Click OK. Move point P so that its x-coordinate is as close to $x = \pi$ as possible. Repeat the above procedure to find your estimate of $\lim\limits_{x_Q \to \pi} \dfrac{f(x_Q) - f(\pi)}{x_Q - \pi}$, or $f'(\pi)$.

Q1 The above limit looks different from the first limit in the activity. How is the notation different from what you saw above? Has the change in notation altered the value of the limit or the average rate?

2. Zoom in on the following graphs at the following points, and make the value of x_Q as close as possible to x_P. (Don't forget to do this on both sides of point P.) You could run into some problems here.



   a. $f_1(x) = |x - 2|$ for $x_P = 2$

   b. $f_2(x) = |x^2 - 6x + 8|$ for $x_P = 4$

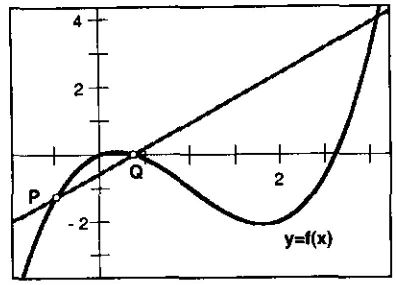
   c. $f_3(x) = \sqrt{x - 1}$ for $x_P = 1$

   d. $f_4(x) = \ln(x)$ for $x_P = 1$

Q2 Are some of the cases in step 2 different from the previous examples? How? What conclusion can you reach? Justify your answers.

Going Off on a Tangent

Name(s): ____________________



You can see what the average rate of change between two points on a function looks like—it's the slope of the secant line between the two points. You have also learned that as one point approaches the other, average rate approaches instantaneous rate (provided that the limit exists). But what does instantaneous rate *look like*? In this activity you will get more acquainted with the derivative and learn how to *see* it in the slope of a very special line.

Sketch and Investigate

1. **Open** the sketch **Tangents.gsp** in the **Exploring Derivatives** folder.

In this sketch there is a function plotted and a line that intersects the function at a point P. This new line is called the *tangent line* because it intersects the function only once in the region near point P. Its slope is the instantaneous rate of change—or derivative—at point P:

$$\textit{tangent's slope} = \textit{instantaneous rate at } P = f'(x_P)$$

So how do you find this line? Let's hold off on that for a bit and look at the line's slope—the derivative—and see how it behaves. Remember, slope is the key!

Be careful here—the grid is not square!

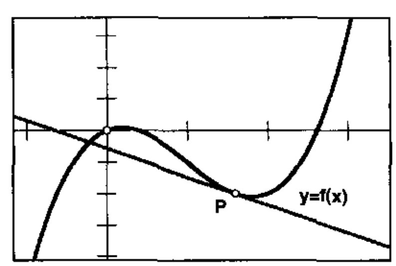
Q1 Move point P as close as possible to $x = -1$. Without using the calculator, estimate $f'(-1)$—the derivative of f at $x = -1$. (*Hint:* What's the slope of the tangent line at $x = -1$?)

Q2 Move point P as close as possible to $x = 0$. Without using the calculator, estimate $f'(0)$—the derivative of f at $x = 0$. (*Hint:* See the previous hint!)

Q3 Move point P as close as possible to $x = 1$. Without using the calculator, estimate $f'(1)$—the derivative of f at $x = 1$. (Sorry, no hint this time.)

If you'd like, you can animate point P by selecting it, then choosing **Animate Point** from the Display menu.

2. Move point P back to about $x = -1$. Drag point P slowly along the function f from left to right. Watch the line's slope carefully so that you can answer some questions.



Q4 For what x-values is the derivative positive? (*Hint:* When is the slope of the tangent line positive?) What can you say about the curve where the derivative is positive?

Q5 For what x-values is the derivative negative? (*Hint:* Look at the hint in Q4 and make up your own hint.) What can you say about the curve where the derivative is negative?

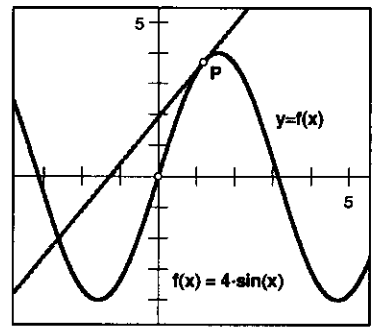
Q6 For what x-values is the derivative 0? What can you say about the curve where the derivative is 0?

Q7 For what value or values of x on the interval from –1 to 3 is the slope of the tangent line the steepest (either positive or negative)? How would you translate this question into the language of derivatives?

3. Go to page 2 of the document. Here the function is $f(x) = 4\sin(x)$.

If you want to recenter your sketch, select the origin and move it to the desired location.

4. Press the *Show Zoom Tools* button and use the *x-scale* slider to change your window to go from -2π to 2π on the x-axis. (You can hide the tools again by pressing the *Hide Zoom Tools* button.)



5. Move point P so that its x-coordinate is around $x = -6$.
6. Move point P slowly along the function to the right until you get to about $x = 6$. As you move the point, watch the tangent line's slope so you can answer the following questions.

Q8 Answer Q4–Q7 for this function. Could you have relied on physical features of the graph to answer these questions quickly? (In other words, could you have answered Q4–Q7 for this function without moving point P?)

There is an interesting relationship between how the slope is increasing or decreasing and whether the tangent line is above or below the curve. Move point P slowly from left to right again on the function, comparing the steepness of the line to its location—above or below the curve.

Q9 When is the slope of the line increasing? Is the tangent line above or below the function when the slope is increasing?

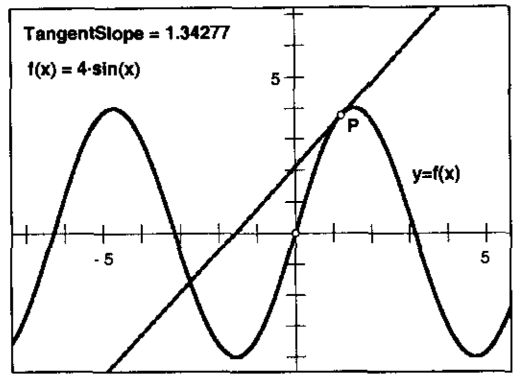
Q10 When is the slope of the line decreasing? Is the tangent line above or below the function when the slope is decreasing?

Q11 Write your conclusion for the relationship between the slope of the tangent line and its location above or below the curve. How would you translate this into a relationship between the derivative and the function's concavity?

Let's check whether or not your conclusion is really true. The derivative is the slope of the tangent line, so an easy way to check is to calculate the slope of the line.

7. Select the tangent line and measure its slope by choosing **Slope** from the Measure menu. Label it *TangentSlope*.

8. Move point P slowly along the function again from left to right and watch the values of the measurement *TangentSlope*.



Q12 Do your answers to Q9–Q10 hold up?

Explore More

Each of the following functions has some interesting problems or characteristics. For each one, change the equation for $f(x)$ by double-clicking on the expression for $f(x)$ and entering in the new expression. Then answer the questions below. If you need to zoom in at a point, press the *Show Zoom Tools* button. Remember that (a, b) represents the point you will zoom in on. To change a or b, double-click on the parameter and enter a new value.

$$f_1(x) = |x - 2|$$

$$f_2(x) = |x^2 - 6x + 8|$$

$$f_3(x) = \sqrt{x - 1}$$

Q1 Where does the derivative not exist for $f_1(x)$ and why? (What happens to the tangent line at that point?)

Q2 Answer Q1 for $f_2(x) = |x^2 - 6x + 8|$.

Q3 Answer Q1 for $f_3(x) = \sqrt{x - 1}$.

Q4 How is the function $f_1(x) = |x - 2|$ different from all the others that you have looked at in this activity, including f_2 and f_3?

Plotting the Derivative

Name(s): ______________________

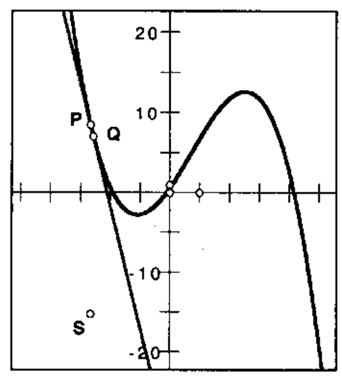
In this activity, you start with the plot of a cubic function, $f(x)$. Your job is to investigate the behavior of the slope of the function—its derivative as defined by the line tangent to the function. The goal of your investigation is to be able to predict and trace the resulting derivative function (the slope of the graph as a function of the x-values).

Sketch and Investigate

1. **Open** the document **PlotDerivative.gsp** in the **Exploring Derivatives** folder. Page 1 shows the plot of a cubic function. Sliders on the left allow you to vary the function.
2. Use the **Line** tool to construct a line with both construction points on the plot. Label the left point P and the right point Q with the **Text** tool.
3. With just point P selected, measure its x-coordinate by choosing **Abscissa (x)** from the Measure menu.

Select the line and choose **Slope** from the Measure menu.

4. Measure the slope of the line and move point Q relatively close to point P.



If the command is not enabled, make sure you have exactly two measurements selected, and nothing else.

5. Select x_P and the slope measurement, *slopePQ*, in that order. Choose **Plot As (x, y)** from the Graph menu. Label this point S.

Q1 *SlopePQ*, or the y-coordinate of point S, represents an approximation of the slope or derivative of $f(x)$ at point P. Explain why it is only an approximation and how you can minimize the approximation error.

To see the plot of the slope or "derivative" function, you can track the behavior of point S as points P and Q move along the graph.

6. With point S selected, choose **Display | Trace Plotted Point.**

If the shape is ragged, try again with the points closer together.

Q2 Using the **Arrow** tool, select just points P and Q, then click on either point and drag them slowly along the curve. What shape does the slope function trace out?

7. Erase the existing traces by pressing the Esc key (twice) or choosing **Erase Traces** from the Display menu.

You can speed them up by pressing the Up Arrow in the Motion Controller.

8. Your trace will be smoother if you animate the points rather than drag them by hand. With both points on the graph selected, choose **Animate Points** from the Display menu. Let the points go far enough to make a complete circuit back to their starting point. Don't press the *Reverse* button.

Q3 Do you notice any difference between the left-to-right trip and the right-to-left trip of the points? If so, why do you suppose this difference exists?

Q4 Press the *Pause* button in the Motion Controller, move the points farther apart. With points P and Q selected, press *Pause* again. Allow the motion to continue for a full cycle. What do you notice about the discrepancy between the two passes of the points?

A full cycle means going all the way to the right, then all the way to the left, and then back to your starting points.

Q5 Pause the animation, move points P and Q as close together as you can, and erase your traces. Again select both points and release the *Pause* button. Allow the motion to continue for a full cycle, then stop. What do you notice this time about the discrepancy between the two passes of the points? (Don't erase the trace this time.)

By using two arbitrary points, you have plotted point S with a y-value of $\frac{f(x_Q)-f(x_P)}{x_Q-x_P}$. By taking point Q close to point P, your y-value approximates $\lim_{Q\to P}\frac{f(x_Q)-f(x_P)}{x_Q-x_P}$, or the derivative at $x = x_P$. Now, if you fix point Q's x-coordinate a set distance, h, from point P's x-coordinate, you'll still be approximating the derivative. You'll just be using another form of the definition and approximating $\lim_{h\to 0}\frac{f(x_P+h)-f(x_P)}{h}$ instead.

9. Select point Q and choose **Split Point From Function Plot** from the Edit menu. (Still, don't erase the trace.)

Click on a measurement or function to enter it into the calculator.

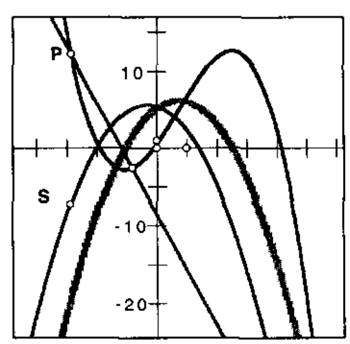
10. Choose **Calculate** from the Measure menu and calculate $x_P + h$. Then calculate $f(x_P + h)$.

11. Plot the point $(x_P + h, f(x_P + h))$ by selecting $x_P + h$, then selecting $f(x_P + h)$, and choosing **Plot as (x, y)** from the Graph menu.

Use the **Text** tool to relabel the merged point Q.

12. Select point Q and the new point and choose **Edit | Merge Points.**

Now you have a secant line and plotted point similar to the ones you constructed by hand in steps 1–6, but with the value h determining the horizontal separation between the two points on the function. This allows you to animate just point P and know that the points on the function will remain separated by the same amount. It also allows you to actually construct the locus of the plotted point S.



13. Turn off tracing for point S (but don't erase the previous trace). With point S still selected, select point P and choose **Construct | Locus.**

14. Experiment with the slider for h to see how it affects the locus of point S.

Q6 What must you do with your slider for h to make the locus match the trace you made in Q5? Explain what you are doing mathematically when you match up the trace and the locus by adjusting h. (When you are done, press the Esc key twice to erase the trace.)

Explore More

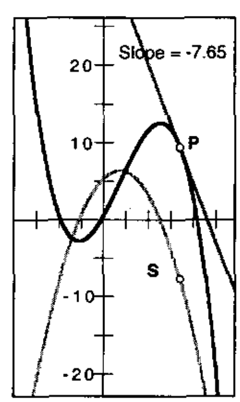
In the activity "Going Off on a Tangent," you used the slope of the tangent line to find the derivative at a point. Could you make a rough sketch of the derivative by tracking how the tangent line's slope changes?

1. Select the locus and point S, then choose **Hide/Show** from the Action Buttons submenu in the Edit menu.
2. Press the *Hide Objects* button. Then adjust any of the sliders *a–d* to make a new cubic function.

Check the status line or the Motion Controller to see if point P is selected. If it isn't, click on the point again.

3. Adjust the h slider as close as possible to $h = 0$ so that you have a good approximation of the tangent line at $x = x_P$. Move point P along the function plot and observe how the slope of this approximate tangent line changes. When is the slope of the line positive? Negative? Zero?
4. Predict what you think the locus of point S will look like and make a little sketch of your prediction in the margin.
5. Press the *Show Objects* button. Compare your prediction with the locus shown. How did you do?

By comparing the slope of the secant line (approximated tangent line) with the location of point S as you drag point P, you can discover the relationship between them. This will help you predict what any function's derivative plot will look like. The following questions will help you discover these relationships. To answer these questions, move point P along the function and focus on the slope of the line and the location of point S.



Q1 When the slope of the line is positive, what can you say about the location of point S? When the slope of the line is negative? When the slope is 0?

Make sure you are looking at the slope of the line and not the function f.

Q2 Drag P slowly from left to right. When the *slope* of the line is increasing, what can you say about the location of point S? When the slope of the line is decreasing?

6. Check your answers by pressing the *Hide Objects* button and adjusting the sliders to create a different function plot. Move point P along the function,

make a prediction for the locus, and sketch it in the margin. Then show the locus and see how you did this time.

7. Once you are satisfied you can predict the graph for these cubic functions, double-click on the expression for $f(x)$ and edit it to $f(x) = ax^4 + bx^3 + cx^2 + dx + e$. Repeat step 4.

8. Adjust the sliders to create a variety of fourth degree polynomial functions. (Make sure to try some negative values for the sliders.) Try to predict the locus in each case.

Q3 Describe the graph that point S sweeps out. Is it consistent with your answers to Q1 and Q2?

Try predicting what the locus will look like for other functions (see below), and then showing the locus to check your predictions.

For each function, describe the locus traced by point S for the functions you have created and check that they are consistent with your answers to Q1 and Q2.

Examples to try:

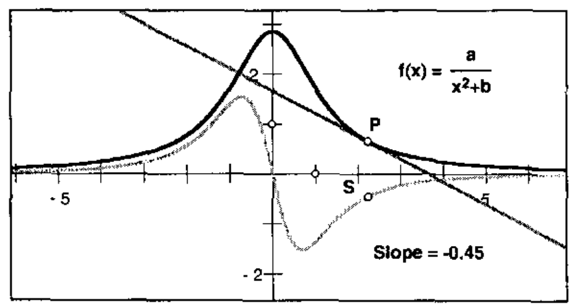
$f_1(x) = a^x$ for $a > 1$

$f_2(x) = a^x$ for $0 < a < 1$

$f_3(x) = a\sin\ (bx)$

$f_4(x) = \dfrac{a}{x^2 + b}$

$f_5(x) = \sqrt{9 - x^2}$



So What's the Function?

Name(s): ____________________

In earlier activities, you created traces or plots of the approximate derivative by finding the slope of the tangent line at a point and then plotting the slope, one point at a time. It would be so much easier and faster if we could create a function that calculates the derivative at every point all at once. Sound impossible? Luckily it isn't, and that is what this activity is all about.

Sketch and Investigate

The derivative of $f(x)$ at $x = x_P$ can be approximated by the ratio $\frac{f(x_P + h) - f(x_P)}{h}$ for h near 0. We can easily do the same thing to approximate the derivative *function*—just use the variable x in place of the constant x_P.

1. **Open** the document **DerFunction.gsp** in the **Exploring Derivatives** folder. On page 1 you will find a linear function $f(x) = bx + c$, its plot, and various sliders.

Q1 You know that the derivative at a point is the function's slope at that point, so for a linear function, what is the derivative at all points?

To check your answer graphically...

2. Choose **Plot New Function** from the Graph menu and for $g(x)$ enter the expression $(f(x + h) - f(x))/h$.

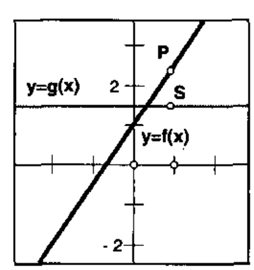
3. Using the **Point** tool, plot a point anywhere on the function $f(x)$. Label this new point P.

Click on the expression for g and then on measurement x_P to enter them into the calculator.

4. With point P selected, measure the x-coordinate for point P by choosing **Measure | Abscissa (x)**. Then choose **Calculate** from the Measure menu and calculate $g(x_P)$.

*Select the measurements x_P and $g(x_P)$, in that order. Then choose **Plot As (x, y)** from the Graph menu.*

5. Plot the point $(x_P, g(x_P))$. Label this new point S with the **Text** tool.



6. Experiment with the sliders for b, c, and h to see how they affect the y-coordinate of point S.

Q2 Show why $g(x)$ is the horizontal line $y = b$ for any h and c by simplifying the expression in step 2 algebraically for $f(x) = bx + c$.

7. Edit $f(x)$ by double-clicking on the expression for f, deleting $bx + c$, and entering $f(x) = ax^2$.

You can also use the button provided to set $h = 0.00001$.

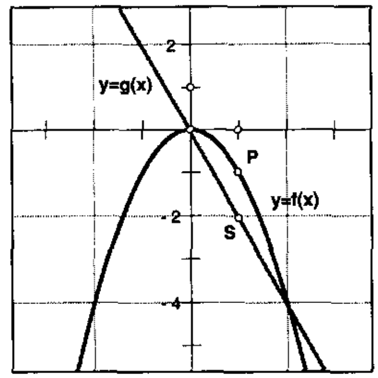
8. Adjust the h slider so that you have a very good approximation for the derivative function.

Q3 Experiment with the a slider; what *kind* of function do the derivative approximations for $f(x) = ax^2$ appear to be?

Q4 Press the *Case: a = 1* button. What function does the derivative approximation function appear to be if $a = 1$? (Write an equation.)

Q5 Examine the values of x_P and $g(x_P)$ as you move point P along the function plot of $f(x)$. What is the relationship between x_P and $g(x_P)$? Do your observations support your answer to Q4?

Q6 Press the other *Case* buttons. In each case, make a table and write down at least three coordinate pairs $(x_P, g(x_P))$. Try to find a relationship between the x-coordinate, x_P, and the y-coordinate, $g(x_P)$, of the derivative approximation. Write your function for $g(x)$. Does your function agree with the basic shape you gave in Q3?



To graphically verify your results, you can plot your function from Q6. Make sure your function matches the function g for *all* values of a!

Q7 Algebraically verify your results in Q6 by simplifying the expression in step 2 for $f(x) = ax^2$ and then calculating its limit as h approaches 0.

Q8 Putting together your work from Q2 and Q7, what do you think the derivative function will be for $f(x) = ax^2 + bx + c$? Check your answer by editing $f(x)$ and then pressing the *Show Derivative* button.

Q9 Experiment with the sliders a, b, and c one at a time. For each one, describe how it affects the plot of f and then how it affects the plot of the derivative of f.

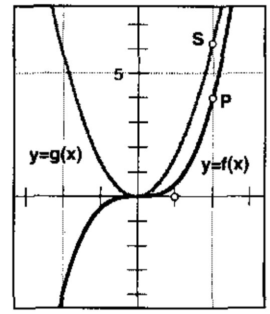
9. Press the *Hide Derivative* button. Then edit $f(x)$ to $ax^3 + bx^2 + cx + d$.

10. Experiment with the sliders that define $f(x)$. Make sure that the h slider is still in the position to give a very good approximation for the derivative function.

Q10 What *kind* of function does the approximation for the derivative appear to be for a cubic function?

11. Press the *Case: a = 1* button to set $a = 1$ and the rest of the sliders to 0. Examine the values of x_P and $g(x_P)$ as you move point P along the plot of $f(x)$.

Q11 Write down at least three coordinate pairs $(x_P, g(x_P))$ and find a relationship between the x-coordinate, x_P, and the y-coordinate, $g(x_P)$, of the derivative approximation. What function does the derivative approximation function appear to be if $a = 1$? (Write an equation.)



Q12 Press the other *Case* buttons. In each case, make a table and write down at least three coordinate pairs $(x_P, g(x_P))$ to find a relationship between the x-coordinate, x_P, and the y-coordinate, $g(x_P)$, of the derivative approximation. Write your function for $g(x)$. Does your function agree with the basic shape you gave in Q10?

Q13 Algebraically verify your results from Q12 by simplifying the expression in step 2 for $f(x) = ax^3$ and calculating its limit as h approaches 0.

Q14 Putting together your work from Q2, Q7, and Q13, what do you think the derivative function will be for $f(x) = ax^3 + bx^2 + cx + d$? Check your answer by editing $f(x)$ and then pressing the *Show Derivative* button.

Q15 Experiment with the sliders a, b, c, and d one at a time. For each one, describe how it affects the plot of f and then how it affects the plot of the derivative of f.

Exploration 1

Finding the derivative algebraically by using the limit definition is much easier for polynomials than for most other functions, such as logarithms, trigonometrics, exponentials, roots, and so on. But the graphical procedure for approximating them is much the same.

1. Edit $f(x)$ to one of the basic functions given below.
2. Make a conjecture about its derivative function by examining the basic shape of the function $g(x)$. Then use the relationship between the x-coordinate, x_P, and the y-coordinate, $g(x_P)$, to predict or figure out the exact equation.
3. Write your function for $g(x)$ and check by pressing the *Show Derivative* button.

Basic Functions:

$f(x) = \sin(x), \cos(x), \text{ or } \tan(x)$

$f(x) = \sqrt{x} \text{ or } x^{(p/q)}$

$f(x) = \dfrac{1}{x} \text{ or } \dfrac{1}{x^2}$

$f(x) = \log(x) \text{ or } \ln(x)$

$f(x) = a^x \text{ for } 0 < a < 1 \text{ or for } a > 1$

Exploration 2

When we were looking for the derivative for the general quadratic and cubic functions, you used your previous work and added the two derivatives together—in other words,

$$(ax^2 + bx + c)' = (ax^2)' + (bx + c)'$$

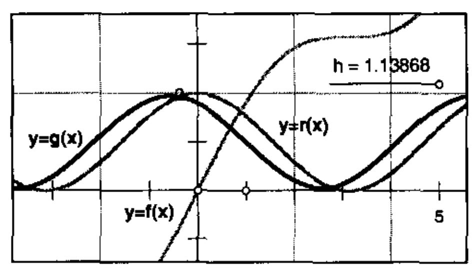
Is it always true that $(p_1 + p_2)' = (p_1)' + (p_2)'$, for any functions p_1 and p_2?

To label the function p_1, double-click on the expression for $p_1(x)$ with your **Text** tool and label it p[1].

1. Go to page 2. Choose **New Function** from the Graph menu to create a new function $p_1(x) = \sin(x)$. Then create another new function $p_2(x) = x$. Do not plot these functions.
2. Edit $f(x)$ to $p_1(x) + p_2(x)$.
3. Calculate the derivative for $p_1(x)$ by selecting it and choosing **Derivative** from the Graph menu.
4. Calculate the derivative for $p_2(x)$.

Click in an empty spot to deselect all objects before you go to the Graph menu.

5. Create a new function $r(x) = p_1'(x) + p_2'(x)$ by this time choosing **Plot New Function** from the Graph menu. Make the resulting plot a new color.
6. Adjust the slider for h so that $g(x)$ is a very good approximation for the derivative of $f(x)$.



Q1 Does the derivative of the sum, $f'(x)$, of two functions appear to be the same as the sum of the two derivatives, $r(x)$?

Investigate other combinations of p_1 and p_2 by editing $f(x)$ and $r(x)$ to answer the following questions:

Q2 Does the derivative of the difference $f'(x)$ of two functions, $f(x) = p_1(x) - p_2(x)$, appear to be the same as the difference of the two derivatives, $r(x) = p_1'(x) - p_2'(x)$?

Q3 Does the derivative of the product $f'(x)$ of two functions, $f(x) = p_1(x) \cdot p_2(x)$, appear to be the same as the product of the two derivatives, $r(x) = p_1'(x) \cdot p_2'(x)$?

Q4 Does the derivative of the quotient $f'(x)$ of two functions, $f(x) = p_1(x)/p_2(x)$, appear to be the same as the quotient of the two derivatives, $r(x) = p_1'(x)/p_2'(x)$?

Derivatives of Exponential Functions

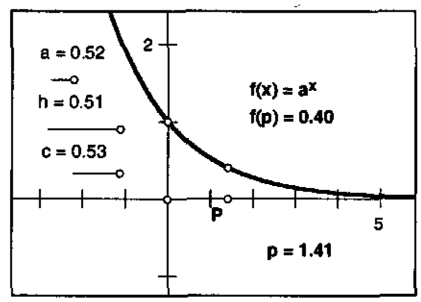
Name(s):____________________________

Consider the family of functions, $f(x) = a^x$ for $a \geq 0$. Does the derivative of f end up being a log or an exponential or does it follow the power rule? In this activity, you will explore these possibilities and see if you can derive a formula for the derivative of an exponential function.

Sketch and Investigate

Start with a doubling function, $f(x) = 2^x$. Sketch its basic graph by hand. Is the graph increasing or decreasing? Concave up or concave down? Draw a rough sketch of what the derivative should look like based on your basic graph's behavior. Any guesses yet on what the derivative might be? Can you eliminate any choices?

This time, instead of sketching a tangent line, you will use the definition of the derivative, $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, and approximate this limit using Sketchpad, thus creating an approximation for the derivative of $f(x) = 2^x$.



1. **Open** the sketch **Exponent.gsp** in the **Exploring Derivatives** folder. Axes and a point P $(p, 0)$ are included, as are three sliders labeled a, c, and h. The measurement p is the x-coordinate of point P.

Choose **Plot New Function** from the Graph menu.

2. Plot a new function $f(x) = a^x$.

Calculate $f(p)$, then select p and $f(p)$, in that order, and choose **Plot as (x, y)** from the Graph menu.

3. Plot the point that corresponds with $x = p$ on $f(x)$.
4. Drag a and observe the behavior of the function for different values of a.

Q1 For the family of functions $\{f(x) = a^x \mid a \geq 0\}$, what values for a give increasing functions? Decreasing functions?

Q2 What happens when $a = 1$? Why? What happens when $a \leq 0$? Why?

Q3 Compare the graph of $f(x) = a^x$ for two values of $a > 1$. How are the graphs the same? How are they different?

Q4 Using the limit definition of the derivative, what is the expression for the derivative of $f(x) = 2^x$?

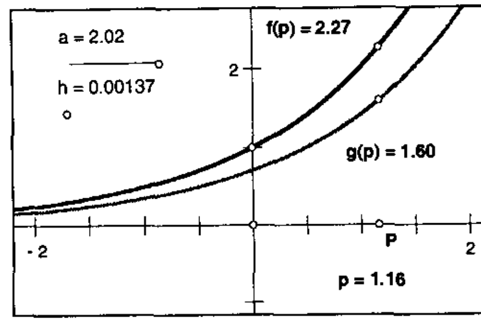
How can you use Sketchpad to model this limit and derivative? (*Hint:* The sketch has more sliders that you have not used. No fair peeking.)

Make g a new color. To label your graphs, click on the plot once with your **Text** tool.

5. Plot $g(x) = \dfrac{a^{(x+h)} - a^x}{h}$.

6. Set the slider for a as close as possible to 2.

7. Set the slider for h as close as you can to 0.



Calculate $g(p)$. Select p and $g(p)$ in that order, then choose **Plot as (x, y)** from the Graph menu.

8. Plot the point that corresponds with $x = p$ on $g(x)$.

Q5 Why is $g(x)$ not the derivative of $f(x) = a^x$? What does $g(x)$ represent?

Q6 How can you use $g(x)$ and Sketchpad to find a good approximation for $f'(x)$?

Q7 Compare $g(x)$ to your basic graphs—what type of function is g? Does g look like the rough sketch you did at the beginning?

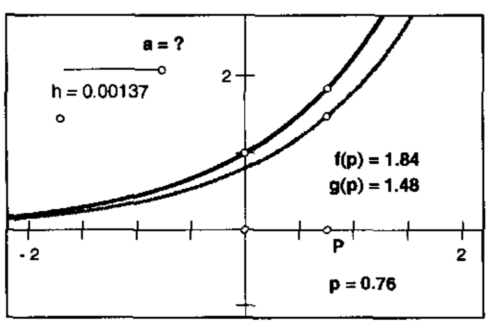
The graphs you have created can be altered by varying the slider for a. Before dragging a, predict what you will see for the "derivative" for various values of a: $a \le 0$, $0 < a < 1$, $a = 1$, and $a > 1$. Once you have your prediction, drag a through various values and see if your predictions are correct.

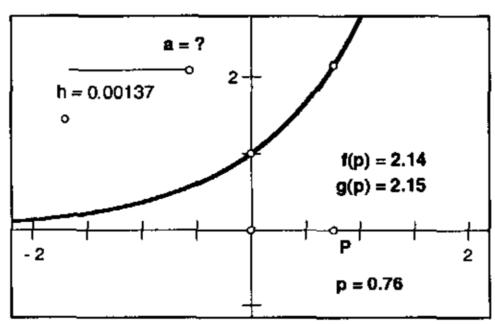
Q8 What happens when $a = 1$? $a \le 0$?

Q9 What happens to our "derivative" when $0 < a < 1$? Why?

Q10 What happens when you increase a to values greater than 1?

In the sketches below, a has been changed until the graphs of f and g appear to coincide. Using the slider for a, figure out when this happens.





Q11 What value of a makes f and g coincide?

That the two plots can coincide suggests that a function can be its own derivative! The number a at which this occurs appears to be close to the value of the number $e \approx 2.7183$. Is e the value of a for which the function $f(x) = a^x$ is its own derivative? Sketchpad cannot prove this to be true, but it can be used to explore the possibility.

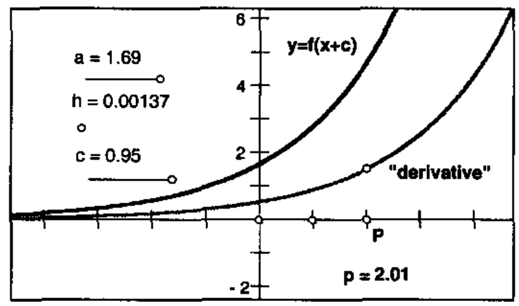
Vary a again. When the graphs of $f(x) = a^x$ and its derivative do not coincide, how do they appear to be related? This time, focus on a possible equation for the graph of the derivative of $f(x) = a^x$. Remember there are translations: $f(x + c)$ and $f(x) + c$, and stretches: $f(cx)$ and $cf(x)$.

This is a hard question, so let's experiment with f and a new slider, c.

Choose **Plot New Function** from the Graph menu. To enter f and c in the calculator, select them with the **Arrow** tool.

9. Plot a new function, $h(x) = f(x + c)$, and hide the graph of f by selecting just the plot of $f(x)$ and choosing **Edit | Action Buttons | Hide/Show.** Then press this new button to hide f.

10. Fix any $a > 1$ and vary c. Can you get your new graph to coincide with the graph of the "derivative"? If so, continue with step 11. If not, go to your next transformation in step 12.

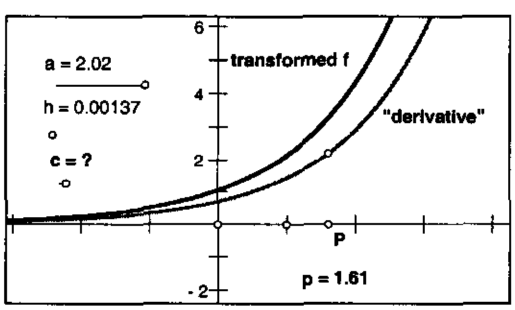


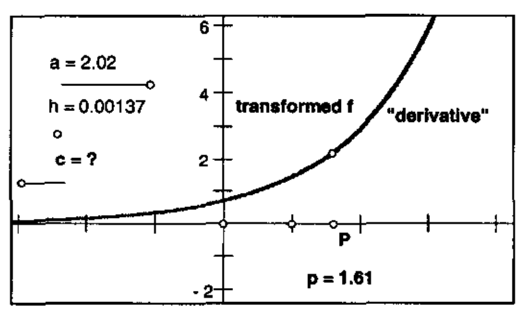
11. Fix any $0 < a < 1$ and vary c. Can you get your new graph to coincide again? If so, then you've found the right transformation. If not, let's go to the next one.

12. Double-click on the transformation expression $h(x) = f(x + c)$—not the plot of h—and change the expression to $f(x) + c$, $f(cx)$, or $cf(x)$.

13. Repeat steps 10–12 until you discover the transformation that works for both intervals: $a > 1$ and $a < 1$.

When you have discovered which transformation works, you will want to find a value for c. What value of c is needed to create the sketch below when $a \approx 2$? Is there any relationship between the value of c and the number 2?





Explore More

Did you discover that the transformation is a vertical stretch, so $f'(x) \approx g(x) \approx c \cdot f(x)$? If you didn't, try steps 9–13 again. If you divide both sides of $g(x) \approx c \cdot f(x)$ by $f(x)$ you end up with $\dfrac{g(x)}{f(x)} \approx c$. But c is a constant! So the ratio of the y-values is a constant for each value of a. (*Note:* This is true for any vertical translation.) In the next few steps, you will use Sketchpad to check this fact. This may also lead to a formula for the derivative of the general exponential function, $f(x) = a^x$.

1. Use the calculator to find the ratio of $\dfrac{a^{(x+h)} - a^x}{h}$ to a^x by using the ratio $\dfrac{g(p)}{f(p)}$.
2. Relabel this ratio *constant.*
3. Fix any $a > 0$. Slide point P along the x-axis to change the value of p.

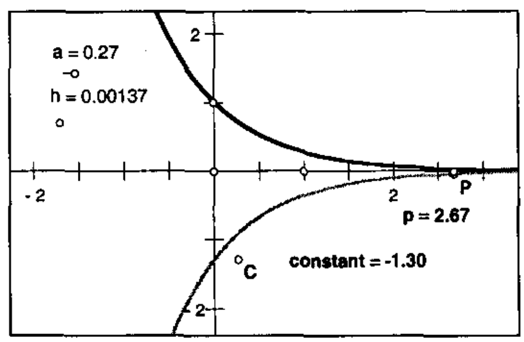
Q1 Does the measurement *constant* change as the value of p changes?

4. Fix point P. Change the value of a using the slider.

Q2 Does the measurement *constant* change as the value of a changes?

It's hard to discern a pattern by just looking at a bunch of numbers. Remember, you are trying to figure out the value for *constant* for different values of a.

So try plotting the values.



5. Select the measurement a and the measurement *constant.* Choose **Plot as (x, y)** from the Graph menu.
6. With just your new point selected, choose **Trace Plotted Point** from the Display menu. Label the plotted point C.

Q3 Fix any $a > 0$. Slide point P along the x-axis to change the value of p. Does point C move as point P changes?

Q4 Fix point P. Change the value of a using the slider. Does point C move as point P changes?

To see the locus of points, you can select point *C* and point *a*—the actual point on the end of slider *a*—in that order. Then choose **Locus** from the Construct menu.

Q5 One of the above movements created a trace or locus of points. Do you recognize the graph? Any ideas now what the derivative of $f(x) = a^x$ is?

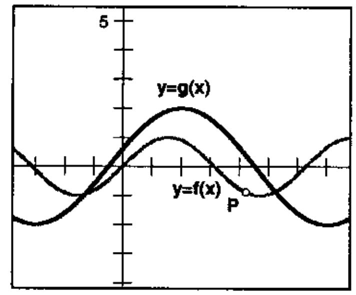
Derivatives and Transformations

Name(s): ____________________

If you know the derivative for a given basic function, can you predict the derivative for a transformation of that basic function? For example, the derivative for $f(x) = \sin(x)$ is $f'(x) = \cos(x)$, so what is the derivative for $f(x) = 2\sin(3x) + 4$? In this activity, you will explore this question graphically.

Sketch and Investigate

There are four basic possible transformations: vertical and horizontal translations, and vertical and horizontal stretches. If you have a basic function $f(x)$, you can represent all of them at once by $g(x) = af(bx - c) + d$. We'll investigate these transformations one at a time to see how each constant affects the plot graphically and how it affects the derivative.



1. **Open** the document **Transformations.gsp** in the **Exploring Derivatives** folder. On page 1 is the function $f(x) = \sin(x)$ and the transformation $g(x) = f(x) + d$.
2. Adjust the slider for d.

Q1 What kind of graphical transformation occurs when you change the value of d? Specifically, what happens if d is negative? Positive? Why does this occur?

Q2 How do you think the derivative is affected by the transformation?

One way to see how the derivative is affected is to examine the slopes of the tangent lines at corresponding points—points P_f and P_g.

3. Press the *Show Tangents* button.

You can animate point P_f instead by selecting point P_f and choosing **Animate Point** from the Display menu.

4. Move point P_f along the function $f(x)$.

Q3 How is the slope of the tangent line to the function $g(x)$ at point P_g related to the slope of the tangent line to the function $f(x)$ at point P_f?

Q4 Write an equation for $g'(x)$ in terms of $f'(x)$.

Select the expression for f' to enter it into the New Function panel.

5. To check your answer to Q4, press the *Show f′* button and choose **Plot New Function** from the Graph menu. Enter your equation from Q4. Then press the *Show g′* button.

Q5 Is the function you plotted in step 5 identical to the plot of g'? If so, great! If not, try again.

Adding d to the function's value is one type of transformation. What happens if a constant is added to, or subtracted from, the x-value first?

6. Go to page 2 of the document. Here we have the transformation $g(x) = f(x - c)$.
7. Adjust the slider for c.

Q6 What kind of graphical transformation occurs when you change the value of c? Specifically, what happens if c is negative? Positive? Why does this occur?

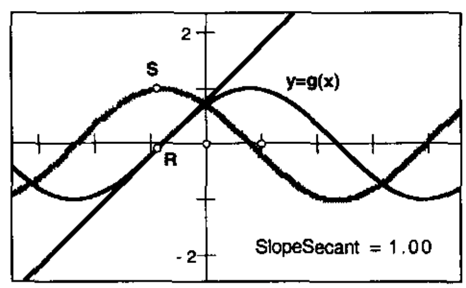
Q7 How do you think the derivative is affected by the transformation?

You can also see how the derivative is affected by tracing a point representing the slope of the secant line between the two points $(x, g(x))$ and $(x + h, g(x + h))$, and then having h approach 0.

8. Choose **Secant Line** from **Custom** tools. Click on point R (on the plot of function g), and then on the expression for $g(x)$.

This tool makes the new point $(x_R + h, g(x_R + h))$ and calculates the measurement *SlopeSecant*, which is the value $\frac{g(x_R + h) - g(x_R)}{h}$.

9. Measure the x-coordinate of point R. With the x-coordinate selected, select the measurement *Slope Secant* and choose **Plot as (x, y)** from the Graph menu. Label this new point S. Make it a new color.
10. Press the *h→0.00001* button and then turn on tracing for point S. With $h = 0.00001$, the trace of point S will be a very good approximation of $g'(x)$.



If you want to eliminate clutter, you can press the *Hide f* button here.

11. Select point R and choose **Animate Point** from the Display menu.

Q8 Press the *Show f′* buttons. Compare your approximations of $g'(x)$ and $f'(x)$.

Q9 Write an equation for $g'(x)$ in terms of $f'(x)$.

12. To check your answer to Q9, choose **Graph | Plot New Function** and enter your equation from Q9. Press the *Show g′* button.

If you don't see a new function when you press the *Show g′* button, then that means the two functions are identical and you are right!

Q10 Is the function you plotted in step 12 identical to the plot of $g'(x)$? If so, great! If not, there are many transformations of the derivative you can try: $f'(x)$, $f'(x) + k$, $f'(x + k)$, $f'(kx)$, $k \cdot f'(x)$, and $k \cdot f'(kx)$. Adjust the slider for k to make your prediction function match your trace. If you can't, then it is time to try the next transformation.

We've now looked at all the transformations possible by adding or subtracting a constant. What kind of transformations occur when you multiply by a constant?

13. Go to page 3 of the document. Here we have the transformation $g(x) = a \cdot f(x)$, with points S and R defined as they were on page 2.
14. Adjust the slider for a.

Q11 What kind of graphical transformation occurs when you change the value of a? Specifically, what happens when a is negative? Positive? Why does this occur?

Q12 How do the graphs compare when a is greater than 1? Positive, but less than 1?

Q13 How do you think the derivative is affected by the transformation?

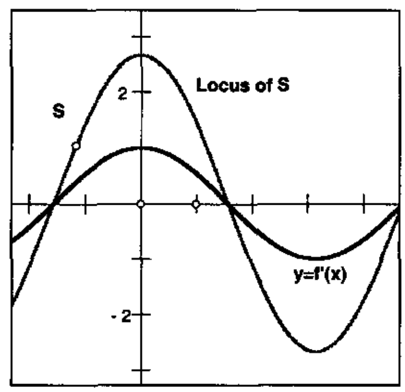
Now you'll use the locus of point S to see how the derivative is affected.

Make sure that your h is still very small!

15. Select points S and R and choose **Locus** from the Construct menu. Make the locus a new color.

Here, you can hide f, g, and the secant line if your sketch is too cluttered.

Q14 *Show* f' and compare the plot of f' and the approximation of g' (the locus). How do your approximations of $g'(x)$ and $f'(x)$ compare?



Q15 Write your prediction for $g'(x)$ in terms of $f'(x)$.

16. To check your answer, plot your prediction, then press the *Show* g' button.

If you don't see a new function when you press the *Show* g' button, then that means the two functions are identical and you are right!

Q16 Is the function you plotted in step 16 identical to the plot of $g'(x)$? If so, great! If not, see Q10 for other functions to try.

The last transformation possible is to multiply the x-value by a constant, but the procedure for finding out how this transforms the derivative is the same.

17. Go to page 4 and do steps 15–16 and answer Q11–Q16 for $g(x) = f(bx)$.

Q17 If you put all the transformations together, you'll get a function like $f(x) = 2\sin(3x) + 4$. Look at your answers for the various transformations and predict what the derivative of this function should be if all the rules you discovered above still work.

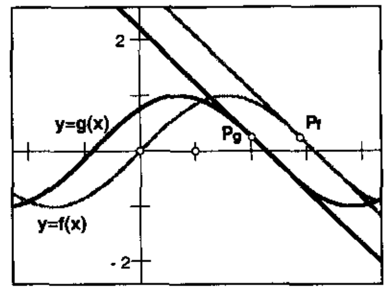
Q18 Now go to page 5 and press the *Transform* button to create a plot of the function $f(x) = 2\sin(3x) + 4$. Use your answer from above to write an expression for the derivative of this function. Then check to see if your prediction is correct.

Explore More

In this section, you'll use tangent lines to figure out how a transformation of $f(x)$ transforms its derivative. When you are using tangent lines to compare the different transformations and their derivatives, the goal will be to get equal slopes—in other words, parallel lines, if possible.

This is a duplicate of page 2 but without the constructions you made there.

1. Go to page 6. Press the *c→0* button and then the *Show Tangents* button.



Q1 Adjust the slider for *c*. How does the tangent line to function $g(x)$ at point P_g move as *c* changes? How does the slope change?

Q2 What are the coordinates of point P_g in terms of the coordinates of point $P_f(x_P, f(x_P))$ and *c*?

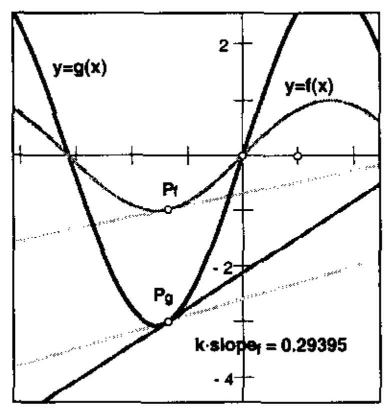
Q3 Using your answers above, what is the equation of the tangent line to function $g(x)$ at point P_g? Check by plotting your new function.

2. Go to page 7. Press the *a→1* button and then the *Show Tangents* button.

Q4 Adjust the slider for *a*. How does the tangent line to function $g(x)$ at point P_g move as *a* changes? How does the slope change?

Q5 What are the coordinates of point P_g in terms of the coordinates of point $P_f(x_P, f(x_P))$ and *a*?

3. To figure out the exact slope of the tangent line, choose **Measure | Calculate** and make the new measurement $k \cdot slope_f$.
4. Then choose **PtSlopeLine** from **Custom** tools. Click on point P_g and then your new measurement.
5. Adjust your slider for *k* until your new line matches the tangent line to function $g(x)$ at point P_g exactly.



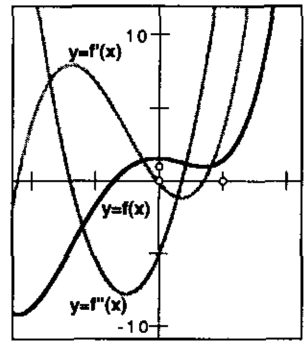
Q6 What value did you get for *k* in order to match the two lines exactly?

Try the same procedure on page 8 to find the *k* for $f(bx)$.

Second Derivatives

Name(s): ______________________

The derivative, $f'(x)$, of a function, $f(x)$, has been a very useful tool. It's given us much information about many things—the behavior of the function $f(x)$, the instantaneous rates of change at points on $f(x)$, and slopes of the function at a point. Since it has been so handy, why not do the process again—take the derivative of the derivative, or the *second derivative* of f. Does it give useful information about the original function as well as information about the first derivative? You will explore this question here.



Sketch and Investigate

You can edit the function f yourself at any time.

1. **Open** the document **SecondDerivative.gsp** in the **Exploring Derivatives** folder. On page 1 is a function $f(x)$, its plot, and various sliders. The sliders allow you to change the function's parameters.

2. You have learned the relationships between a function and its derivative. Press the *Show* f' button to refresh your memory. Then press the *Hide* f and *Hide* f' button and then the *Show* f'' button.

You can also adjust the measurement for c with the appropriate slider.

3. Make the second derivative a constant function by pressing the $a \to 0$ and $b \to 0$ buttons.

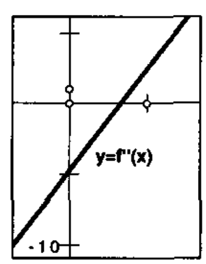
Q1 The function f'' is the first derivative of f', so what should the plot of f' look like if f'' is a constant? Check your answer by pressing the *Show* f' button. Were you right?

Q2 Adjust the slider for c so that $c < 0$, then adjust the slider so that $c > 0$. Describe how this affects the first derivative $f'(x)$.

4. Press the *Hide* f' and then the *Show* f buttons. Adjust the slider for c so that $c < 0$, then adjust the slider so that $c > 0$.

Q3 Describe how adjusting the slider for c affects the function $f(x)$.

Now let's look at what happens if $f''(x)$ is a linear function.



5. Make the second derivative an *increasing* linear function by adjusting the slider for b.

6. Press the *Show* f' button and then adjust the slider for b to make the second derivative a decreasing linear function.

Q4 Describe how adjusting the slider for b affects the first derivative.

Q5 When $f''(x) = 0$, or crosses the x-axis, what can you say about the plot of the first derivative? (Check both cases—$f''(x)$ increasing and decreasing.)

Q6 What is the significance of the root of $f''(x)$ for the function $f(x)$? Check your answer by trying different values of c.

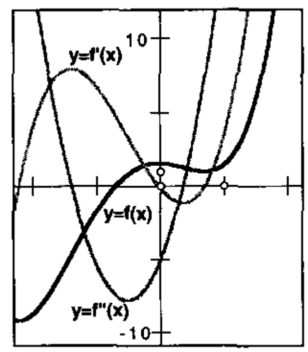
Do the qualities you discovered hold for all functions? Let's look at one more case, where $f''(x)$ is a quadratic.

7. Hide $f(x)$ and $f'(x)$. Use the sliders for a, b, and c to create a new plot of $f''(x)$ where the function is always positive, but not constant.

Q7 Predict what you think will be true about $f(x)$ and $f'(x)$ in this case. Check your answers by showing f and f'. How did you do?

8. Hide $f(x)$and $f'(x)$ again. This time, use the sliders for a, b, and c to create a new plot of $f''(x)$ where the function is always negative.

Q8 Predict what you think will be true about $f(x)$ and $f'(x)$ in this case. Check your answers by showing f and f'. How did you do?



Q9 Make a chart of how $f''(x)$ affects both $f(x)$ and $f'(x)$. Where $f''(x)$ is negative, what happens to $f'(x)$ and $f(x)$? Where $f''(x)$ is positive, what happens to $f'(x)$ and $f(x)$? Where $f''(x)$ is 0, what happens to $f'(x)$ and $f(x)$?

It is easier to check your conclusions if you look at just two functions at a time.

9. Check your conclusions in Q9 by creating a new plot of $f''(x)$ where the function is sometimes negative and sometimes positive. How did you do?

On page 2 of the document, all three plots have the same color. Using what you learned above, can you identify which is which without using the buttons?

10. Predict which plot is $f''(x)$, $f(x)$, and $f'(x)$. Check your answer by pressing the *Show* buttons.

11. Press all three *Hide* buttons to hide the answers. Then, press the *Randomize Sliders* button to change the values of all the sliders. Press the button again to stop. Identify each of the plots again.

There is another relationship between the three functions to see. This one has to do with the tangent line to $f(x)$ at a point. You have seen that the tangent line's slope is the derivative at the point of tangency. But its position—above or below the function—also tells something important.

12. Go back to page 1. Use the *Hide/Show* buttons to make sure only $f(x)$ is showing.

13. Choose **Tangent Line** from **Custom** tools. Click anywhere on the plot of $f(x)$ and then on the expression for $f(x)$ to create a tangent line.

Q10 Show $f''(x)$ and then move your new point along the plot. When $f''(x)$ is negative, what can you say about the tangent line?

Q11 What happens to the tangent line where $f''(x)$ is positive? What happens at the points where $f''(x)$ changes from positive to negative?

Exploration 1

You can also use the limit of the secant line's slope.

In this section, you'll build the second derivative's plot using the limit of the difference quotient just like you built the first derivative's plot—the only difference is that $f'(x)$ is used in place of $f(x)$.

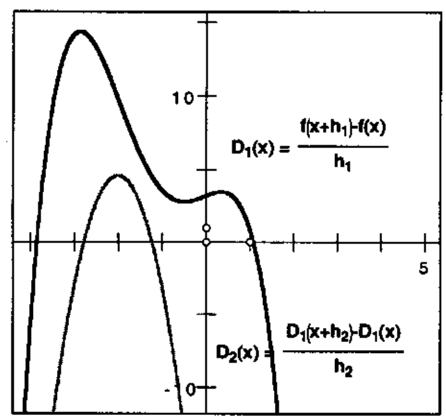
1. Go to page 3. On this page there are two extra sliders that adjust the measurements h_1 and h_2.

Enter any subscripts in brackets. So for D_1 you would enter D[1].

2. Choose **Graph | New Function** and enter the difference quotient for the first derivative: $(f(x + h_1) - f(x))/h_1$. Label this function D_1.

You can check your approximation by plotting your new function and then pressing the *Show f'* button.

3. Adjust the slider for h_1 so that the function $D_1(x)$ is a good approximation for $f'(x)$.



Now if you build the difference quotient using $D_1(x)$, this new function can be used to approximate the second derivative.

4. Choose **Plot New Function** from the Graph menu and enter the expression for the approximation of the second derivative: $(D_1(x + h_2) - D_1(x))/h_2$. Label this function D_2.

5. Press the *Show f''* button and then adjust the slider for h_2 so that the function $D_2(x)$ is a good graphical approximation for $f''(x)$.

6. To see if $D_2(x)$ is a good approximation numerically, choose the **Point** tool, create a point on the x-axis, and label it P. Measure the x-coordinate of point P, then choose **Calculate** from the Measure menu and enter the expression for the error: $f''(x_P) - D_2(x_P)$.

Q1 Were you able to make h_2 small enough so that the error was at most 0.01? (If not, drag the unit point on the x-axis to zoom in a bit).

Now that you have both difference quotients built, you can adjust any of the sliders a, b, c, or d to change the function $f(x)$ or you can double-click on $f(x)$ and enter any other function you'd like to try and the difference quotients will change dynamically as well.

Exploration 2

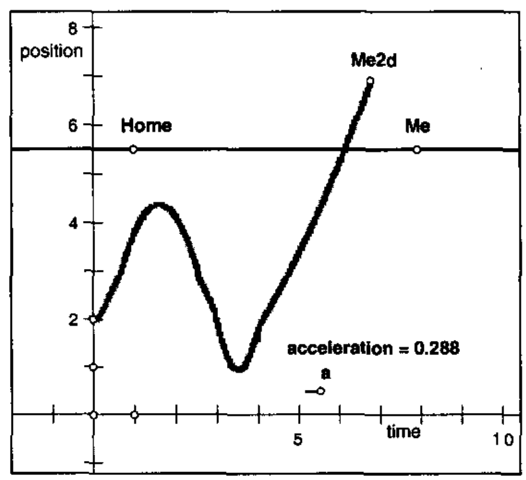
Go to page 4. This page has the plot of the second derivative of a function f, and its equation, $f''(x) = d + a\sin(bx + c)$. When you open the page, $b = 1$, and c, d, e, and $f = 0$. Show the plots of $f(x)$ and $f'(x)$. Probably no surprises there. Hide $f(x)$ and $f'(x)$. Adjust the slider for d so that $d \neq 0$ and predict what you'll see when you look at $f(x)$ and $f'(x)$. Show $f(x)$ and $f'(x)$. Surprised? What is the surprise and why is this so? (You may want to adjust d while $f(x)$ and $f'(x)$ are showing to answer this.)

Try this as well: On page 5, plot the functions $q(x) = x^2$ and $r(x) = \cos(x)$. What do you think the function $q(x) + r(x)$ looks like? Plot the combined function and see what happens. How does the second derivative explain the resulting graph? (Use the **Derivative** command twice to find the first and second derivatives of $q(x) + r(x)$ to visualize the answer.)

Exploration 3

On page 6 of the document, there is a sketch much like the one in "Visualizing Change: Velocity." On this page, you are able to control the acceleration of a point—the second derivative of position. If you set the acceleration to 0 by pressing the *a*→*0* button, then the sketch works just like the one in that activity. (You would use the velocity slider v_c.)

See "Visualizing Change: Velocity" for more information.



Here you can control the acceleration by adjusting the *a* slider. To try this out, set your initial velocity with the v_c slider. Then press the *Start Motion* button and adjust only the *a* slider. The other velocity slider, labeled *velocity*, will change with the acceleration.

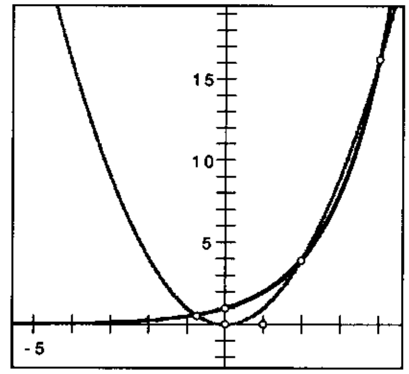
What happens to the velocity if you maintain a positive acceleration? What if you maintain a negative acceleration? Can point *Me* move in the positive direction when the acceleration is negative?

Try various actions with either slider to see how the position trace moves. Then you can try to match Path 1 or Path 2.

Newton's Method

Name(s): ____________________

While you can solve many equations with familiar algebraic techniques, there are also quite a few that you can't solve algebraically: $2^x = x^2$, for instance. From the plot, it looks like there are three solutions to this equation. The ones in the first quadrant you could get by guessing and checking, but probably not the one in the second quadrant. In this activity, you will learn an algorithm for solving equations using derivatives.



Sketch and Investigate

1. **Open** the document **Newton.gsp** in the **Exploring Derivatives** folder. On page 1 you will find the function $f(x) = 2^x - x^2$.

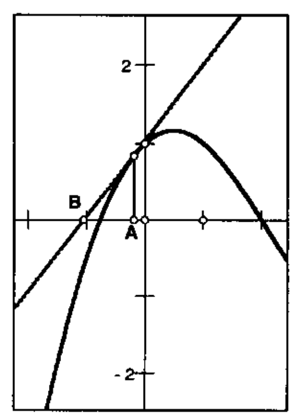
Q1 Why is solving the equation $2^x = x^2$ equivalent to finding the zeroes for $f(x) = 2^x - x^2$?

You have seen that a tangent line to a function at a point can be used to approximate a differentiable function on a small interval around that point. Going further with this idea, you can use the x-intercept of a tangent line to approximate the x-intercept of a function. First we'll see how this works and then you'll make a tool to do the process.

To start the process, we choose a starting point on the x-axis, called a *seed,* and find the point on the function plot that corresponds to that x-value.

To measure the x-coordinate, choose **Measure | Abscissa (x)**.

2. With the **Point** tool, construct a point on the x-axis close, but not too close, to the zero of $f(x)$ on the left side of the origin. Measure the new point's x-coordinate.



3. Calculate $f(x_A)$ by choosing **Calculate** from the Measure menu and click on the expression for f and then on the measurement x_A.

4. Plot the point $(x_A, f(x_A))$ by selecting measurement x_A, then $f(x_A)$, and choosing **Plot As (x, y)** from the Graph menu.

The next step is to construct a tangent line to $f(x)$ at the point $(x_A, f(x_A))$.

You can also click on the point $(x_A, f(x_A))$ with this tool.

5. Choose **Tangent Line** from **Custom** tools and click on point A and then the expression for $f(x)$.

We now need to find where this tangent intersects the x-axis. This intersection point, or *root* of the tangent line, represents the first approximation for the zero of the function.

6. With the **Arrow** tool, construct the intersection of the tangent line with the x-axis by clicking on that spot. Label this point B.

Select point B with the **Arrow** tool and choose **Abscissa (x)** from the Measure menu.

7. Measure the x-coordinate of point B.

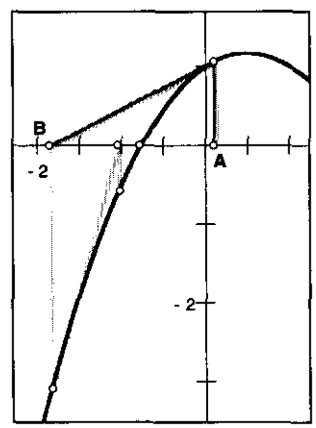
Our goal here is to approximate the zero of the function as closely as possible, so we want to repeat this process until the tangent line's root, point B, is indistinguishable from the function's root.

8. Select point A and point $(x_A, f(x_A))$. Then choose **Segment** from the Construct menu. Go to the Display menu. Make the segment a different color from the Color submenu, and choose **Thick** from the Line Width submenu.

To hide the line, select it and choose **Hide Line** from the Display menu.

9. Hide the tangent line and construct a line segment between the point $(x_A, f(x_A))$ and point B.

10. Select point A and then point B. Choose **Edit | Action Buttons | Movement.** On the Move panel, set speed to **instant**.



11. Select both line segments, then turn tracing on by choosing **Display | Trace Segments.**

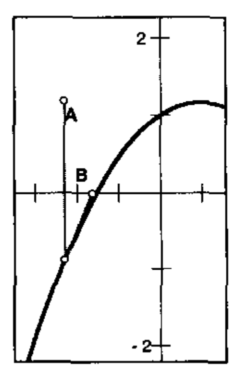
12. Press the *Move A→B* button, and you should see the second iteration. Using the *Move* button, continue "zeroing in" on the root until the x-coordinate of point B is constant to the hundred-thousandths place.

Q2 What final value did you find for the measurement x_B?

Hint: If x_B is a root, what is the value of $f(x_B)$ supposed to be?

Q3 Check your answer by calculating $f(x_B)$. What was your approximation error?

The disadvantage of using a *Move* button and a trace is that if you move your original points, the traces do not move with them. So it is time for a tool.



13. Drag point A out to its original location (which is marked by a segment trace). Turn off tracing for the segments and erase all traces by pressing the Esc key a couple of times.

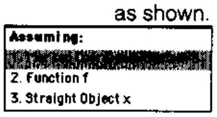
14. With just point A selected, choose **Split Point From Axis** from the Edit menu.

15. To create the tool, select the givens: point A, the expression for $f(x)$, and the x-axis. Select the results: point B, point $(x_A, f(x_A))$, both segments, and the measurement x_B. Then choose **Create New Tool**

from **Custom** tools. Name it **Newton** and check Show Script View. Click OK.

16. Click in an empty spot of the sketch to deselect all objects. Select point *A* and the *x*-axis, and choose **Edit | Merge Point To Axis.**

After you're done, Function f and Straight Object x will be assumed as shown.



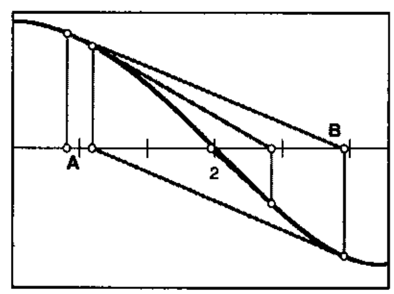
17. In the tool's Script View, double-click given Function f and check Automatically Match Sketch Object. Click OK. Do this again for given Straight Object x.

Now you are ready to use the new tool. If point *B* is too close to the root or point *A* to be distinct, use the **Arrow** tool to move point *A* until point *B* is distinct.

18. Choose **Newton** from **Custom** tools and click on point *B*. The tool will construct a second iteration.

19. Using the tool again, click on the intersection point of the new tangent segment with the *x*-axis to construct a third iteration.

20. Continue using the tool until you have at least five iterations. If at any stage an intersection is not distinct, adjust point *A*'s location before using the tool again.

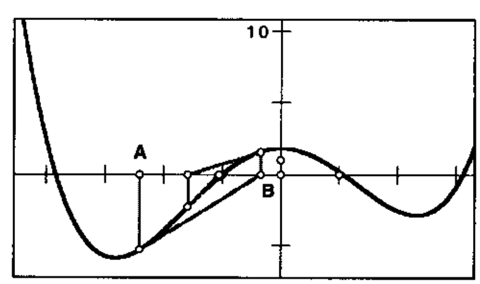


Q4 Adjust point *A* (the seed) so that it's fairly close to a root of the function. What can you say about the convergence of Newton's Method when the seed is close to a root?

Q5 Move the seed value so it's to the left of the leftmost root or to the right of the rightmost root. What can you say about the convergence of Newton's Method in these situations?

Q6 Try some different positions for the seed value between the various roots. What can you say about the stability and convergence of the method under these conditions?

21. Go to page 2, where there is a fourth-degree polynomial. Repeat Q4–Q6 with this function and see if your previous answers hold up.



Q7 Adjust your sliders so that the function has only a single root, but also has a minimum or maximum that comes close to the *x*-axis without touching it. Notice what happens to the iterations as you move the seed value to different positions near this minimum or maximum. What can you conclude about the stability of Newton's Method in this situation?

Q8 Can you think of a way to predict how well Newton's Method will do with different seed values, depending on the shape of the graph? Describe your predictions as clearly as you can.

Explore More

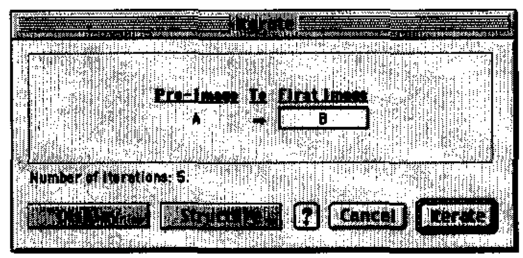
On page 3 of the document, another function is plotted, here with the first step of Newton's Method done. This time you'll use Sketchpad's **Iterate** command to create the successive approximations.

1. Select point A and choose **Iterate** from the Transform menu. Click on point B. Uncheck Tabulate Iterated Values in the Structure pop-up menu. Press the plus (+) key twice to increase the number of iterations to 5. Click **Iterate**.

A good point to click on is the point $(x_B, f(x_B))$.

Select the iterated image and press the plus (+) key to increase the number of iterations.

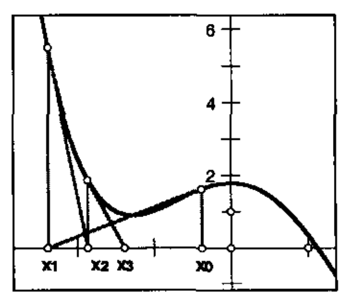
2. Select the iterated image of point A so that only the points are selected. Choose **Terminal Point** from the Transform menu. Measure the coordinates of this point. This is the value found by Newton's Method after the number of iterations you have set. You can increase the number of iterations at any time, and this point will remain at the last iteration.



Q1 Continue your explorations on pages 4, 5, and 6. You only need to move point A. In what situations does the algorithm not bring you to an x-intercept? Explain in your own words why this happens.

Q2 If point A's x-coordinate is labeled x_0, what is the equation for the first tangent line you constructed?

Q3 If point B's x-coordinate is labeled x_1, solve your equation in Q2 to find a formula for finding x_1 in terms of x_0 and the function $f(x)$.



Q4 For the next iteration, the algorithm starts over and now uses x_1, so write a formula for finding x_2 in terms of x_1 and $f(x)$.

Q5 Generalize your work above and write a formula for finding x_{n+1} in terms of x_n and $f(x)$.